The Arithmetic of Growth: Methods of Calculation

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ABSTRACT: This is a tutorial on the relations between population data and the rates of growth that are calculated from the data. For the calculation of rates of growth, discrete and continuous compounding will be compared so that the reader can see the reasons for using the mathematics of continuous compounding, which is the mathematics of exponential growth. Some properties of exponential growth are developed. Semi-logarithmic graphs will be discussed as a device for representing the size of growing populations and for analyzing the nature of the growth. Illustrative examples will be worked out in order to emphasize applications and utility.

DEFINITIONS

If a quantity (such as a population) changes (increases or decreases) by a *fixed amount per unit time* (for example, ten units per year), the quantity is said to be changing *linearly* or *arithmetically*. If the quantity is increasing, we have *growth*; if it is decreasing, we have *decay*. An example would be a population that increases (or decreases) by 100 persons each year.

If a quantity changes (increases or decreases) by a *fixed fraction per unit time*, (for example, ten percent per year), the quantity is said to be changing *exponentially* or *geometrically*. An example would be a population that increases (or decreases) by five percent each month.

If a quantity is always increasing (or decreasing), but the changes do not have the regularity of either linear or exponential change, the growth (or decay) is said to be monotonic or continuous. Linear growth or decay,

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and exponential growth or decay are monotonic. Monotonic growth or decay need not be either linear or exponential.

We want to discuss exponential changes, (growth and decay).

We will define the symbol k to be the fractional change per unit time and the symbol R to be the percent change per unit time which we will call the growth rate. Thus

$$R = 100\ k \tag{1}$$

If $k = +\ 0.03$ per year, then we are dealing with a growth rate of $R = 3$ percent per year.

If k is positive and constant, one has exponential growth; if k is negative and constant, one has exponential decay.

It is unfortunate that the term "exponential growth" is interpreted by some to be a rare and exotic form of growth that is different from the "steady growth" which is so often in the news. We will define "steady growth" to be synonymous with "exponential growth" and "geometric growth."

It is incorrect to use the term "logarithmic growth" to describe exponential growth. The book, *The Logarithmic Century*, (Lapp, 1973) should have been called *The Exponential Century*. It shows many examples of quantities that have been growing exponentially for long periods of time.

THE GROWTH OF POPULATIONS

Populations tend to change exponentially. This can be seen from the units that are used to express birth and death rates. The world birth rate is approximately 27 per thousand each year and the death rate is approximately 10 per thousand each year. The "27 per thousand" expresses the fractional change and the "each year" converts this to the fractional change per unit time. The difference between these two numbers is the increase of 17 per thousand each year, or 1.7 per hundred each year. This means that the growth rate is $R = 1.7$ percent per year, or $k = +\ 0.017$ per year.

"Steady growth" occurs in a period of time if the value of k is positive and constant throughout the period. Very often one deals with growth that is continuous but which is characterized by a changing growth rate. This can be represented in two ways.

a) If we have a changing growth rate during a period of time, the growth may be represented by an average growth rate which is characterized by a constant value of k, or

b) we can tabulate a series of values of k as k changes with time throughout the period. If k is positive and is increasing with time, we have growth that is faster than exponential, and if k is positive but decreasing with time, we have growth that is slower than exponential. If k decreases to zero and becomes negative, the changes switch from growth to decay.

In addition to populations, things that tend to grow exponentially include: money in an interest-bearing savings account, the cost of living, the number of fission events that take place with each generation of neutrons in a nuclear explosion, the number of pages of articles published annually in scientific journals, and the number of kilometers of highway in the United States (Bartlett, 1969). Things that tend to decay exponentially include the value of the dollar, the number of undecayed radioactive atoms in a sample, the amplitude of vibrations of oscillating objects, and the charge on a capacitor that is discharging through a resistor.

EXAMPLE NO.1

Here are recent data for the population of the United States

1990	248.71 million
1980	226.55 million
Increase in the 1980s	22.16 million

The average of the starting and ending populations of the decade is,

$$(248.71 + 226.55) / 2 = 237.63 \text{ million}$$

There are three ways we can express the fractional increase in the decade.

$$(22.16 / 226.55) = 0.0978 \text{ or } 9.78\%$$
$$(22.16 / 237.63) = 0.0933 \text{ or } 9.33\%$$
$$(22.16 / 248.71) = 0.0891 \text{ or } 8.91\%$$

These results can be expressed by saying that the population increase in the decade was 9.78% of the population at the start of the decade; the increase was 9.33% of the average of the populations at the start and the end of the decade; or the increase was 8.91% of the population at the end of the decade. We can divide these numbers by the ten years of the decade to get 0.978%, 0.933%, and 0.891%. Which, if any, of these numbers is the "average annual growth rate" for the decade ? To answer this question, we must examine the arithmetic of steady (exponential) growth.

DISCRETE COMPOUNDING

The arithmetic of population growth is the same as the arithmetic of the growth of money in a savings account because of the compound interest that is added to the account. (After one of my talks on this subject, a banker in the audience said to me that he was completely familiar with the growth of money due to the arithmetic of compound interest, but he had never realized that this arithmetic also applied to the growth of populations.)

If one leaves money untouched in a savings account, the interest is added at regular intervals and the interest is calculated as a fixed fraction (say 5%) of the money in the account. Thus the fractional growth rate, k, of the money in the account is a constant 0.05 per year, (5% per year). This guarantees that the money in the account will grow steadily (exponentially). The speed with which the money grows depends on two things: the interest rate and the frequency with which the interest calculations are made. The importance of the interest rate is obvious; the importance of the frequency of compounding (calculating the interest) is smaller and is less obvious.

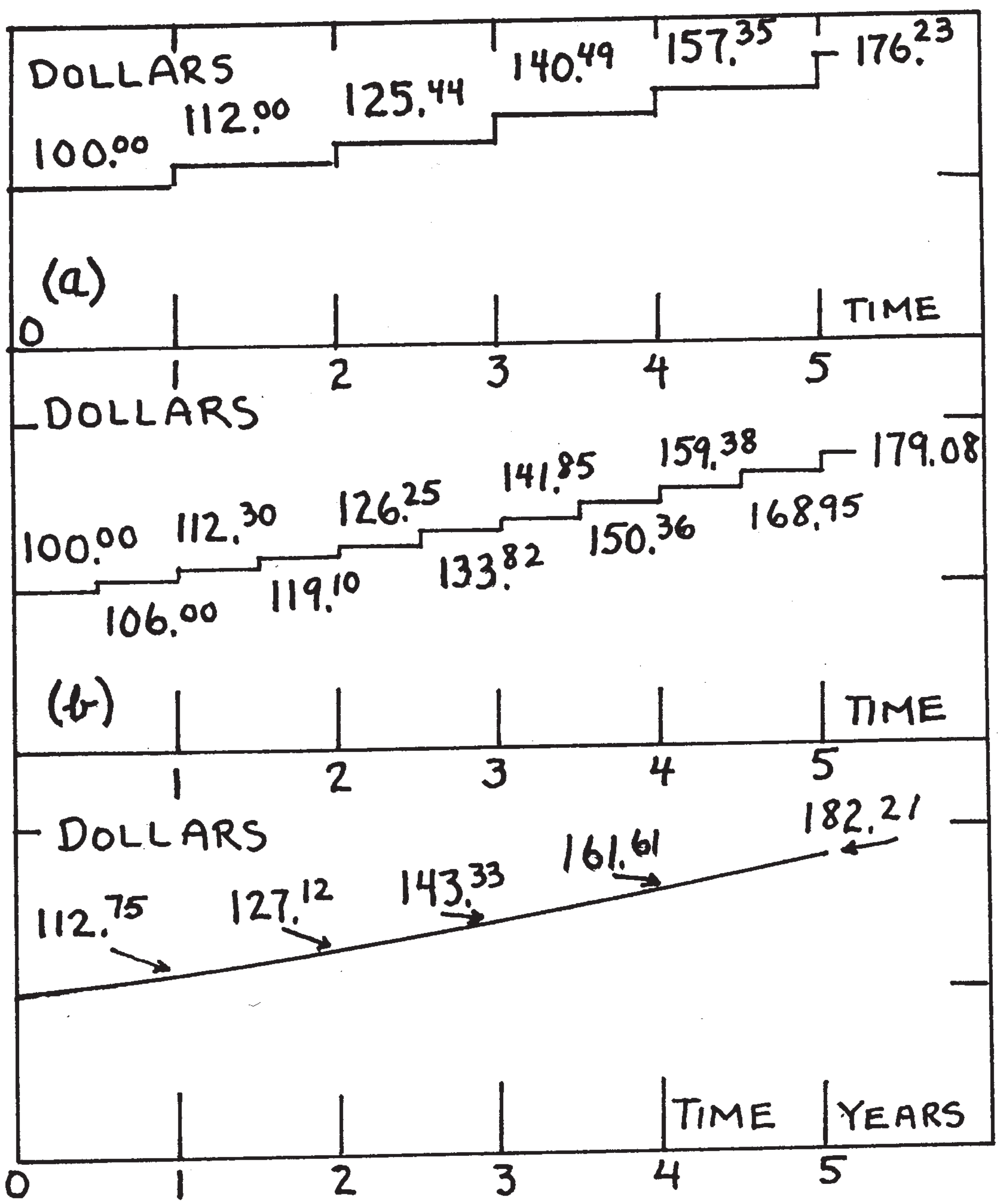
The effect of the frequency of compounding is illustrated in Fig.1. The figure deals with $100 that is placed in a savings account for five years at an annual interest rate of 12%. In Fig.1(a) we see the way in which the dollars increase when the 12% interest is calculated annually at the end of each year. The number of dollars increases in a step fashion, and the dollars in one step are found by multiplying the dollars on the previous step by 1.12. The compounding done at the end of the fifth year leaves $176.23 in the account. In Fig.1(b) we start with the same $100, but the 12% annual interest is compounded now as 6% twice each year. There are now twice as many steps, and the dollars in one step are found by multiplying the dollars on the previous step by 1.06 . The compounding done at the end of the fifth year leaves $179.08 in the account. The 12% could be compounded as 1% every month. In this case, the amount at the end of the fifth year is $181.67. The 12% could be compounded at the daily rate of (12 / 365)% to give $182.19 at the end of five years.

If we want to apply calculations of this type to the calculation of the growth of populations, we see at least three problems.

(1) Populations do not grow stepwise as do the dollars in Figs.1(a) and (b);

(2) The size of the account at the end of five years depends on the frequency of compounding, yet we do not know what frequency of compounding would be appropriate for use in our population calculations;

FIGURE 1. Part (a) shows the growth of $100 at an annual interest rate of 12%, compounded once a year for five years. Part (b) shows the growth of $100 at an annual interest rate of 12% compounded twice a year for five years. Part (c) shows the growth of $100 at an annual interest rate of 12% compounded continuously for five years. Part (c) is the graph of steady or exponential growth.



(3) It would be difficult to calculate the population increase in an interval such as 3.67 years if we were compounding annually or semi-annually.

CONTINUOUS COMPOUNDING

As one can see from Fig.1(a) and (b), the more frequent the compounding, the smaller are the steps in the graph that represents the growth. In the limit, we can compound continuously, and the steps become so small that they merge into a smooth curve. The equation for this curve is

(2) $$N_2 = N_1 e^{k(t_2 - t_1)}$$

In this equation,

N_1 is the size of the growing quantity at time t_1,
N_2 is the size of the growing quantity at time t_2,
k is the fractional increase in N per unit time, and
$e = 2.71828 \ldots$ which is the base of natural logarithms.

This equation is most often written in the form

(3) $$N = N_o e^{kt}$$

In this form, the growing quantity has the size N at the time t, and it had the size $N = N_o$ at the time $t = 0$. The quantity t is a time interval.

It is important that the units of time in k and t be the same. If k has the units *per month* then the units of t must be *months*. Figure 1(c) shows the growth of \$100 in a savings account at an annual interest rate of 12% compounded continuously for five years. It is calculated from Eq.3,

$$N = 100\, e^{0.12t}$$

where t is the time in years. At the end of five years, t has the value 5, and N is calculated to have the value \$182.21.

To do these calculations requires a small hand-held "scientific calculator" that has keys for the functions used by scientists and engineers. In particular the calculator should have keys labeled "e^x", "y^x", "ln x" (or "ln"), and "log x" (or "log"). These calculators can be purchased for as little as \$20.

Continuous compounding eliminates the three problems that arise with discrete compounding, so we will use the arithmetic of continuous compounding to describe the exponential, or steady, growth of populations.

THE AVERAGE GROWTH RATE

If we know the population N_o at the start of an interval of time, and the population N at the end of the interval, we can define the average rate of growth in the interval as that rate of growth, and the corresponding constant value of k, that lets a quantity of size N_o grow to the size N in the given time interval.

If we know N and N_o we can calculate the average growth rate. To do this, we must solve Eq.3 for k.

$$k = (1 / t) \ln(N / N_o) \tag{4}$$

In this equation, "$\ln(N / N_o)$" means the *natural logarithm* of the quotient N divided by N_o. We can rearrange Eq.4 to find the time for N to grow from N_o to N for a given k.

$$t = (1 / k) \ln(N / N_o) \tag{5}$$

BACK TO EXAMPLE NO.1

We will use Eq.4 to calculate the average growth rate of the population of the United States in the time interval t = 10 years between 1980 and 1990.

$$k = (1 / 10) \ln (248.71 / 226.55)$$
$$k = (1 / 10) \ln (1.09781 \ldots)$$
$$k = (1 / 10) \times 0.0933218 \ldots$$
$$k = 0.00933218 \ldots \text{ so that } R = 0.933 \ldots \%\ \text{year}^{-1}$$

Thus, a steady growth rate of 0.933 . . . % per year, will result in a population growing from 226.55 million to 248.71 million in ten years. To three significant figures, the average growth rate happens to equal one of the three growth rates that were calculated earlier.

For linear growth in an interval of time, the average of the populations at the start and end of the interval is the average population during the interval. For exponential growth, the average population in an interval is a little less than the average of the populations at the start and end of the interval. For steady growth for the United States from 1980 to 1990, the average population was 237.46 million.

The formula for calculating the average population, as well as the keystrokes needed to make some of these calculations with a scientific calculator are given in the Appendix.

THE DOUBLING TIME

If a quantity is growing 5% per year, its size is increasing by a fixed fraction (5%) in a fixed length of time (one year), and this is true no matter where one is on the growth curve. Indeed, this is the condition that defines steady growth. It then follows that a longer fixed length off time is required for the growing quantity to increase its size by 100%, which is a doubling of its size. This longer time is called the *doubling time* and it is represented by T_2. The doubling time is the time required for N to grow from its initial size N_o to the size 2 N_o. From Eq.3,

$$2\ N_o = N_o\ e^{(k\ T_2)}$$

$$2 = e^{(k\ T_2)}$$

If we take the natural logarithm of both sides,

$$\ln 2 = k\ T_2$$

(6)
$$\begin{aligned} T_2 &= (\ln 2)/\ k \\ &= 0.693\ldots/\ k, \text{ and since } R = 100\ k, \\ &= 100(.693\ldots)\ /\ R \\ &= 69.3/R \approx 70\ /\ R \end{aligned}$$

Equation (6) is known as "The Law of 70." The equalities of Eq.6 are exact for continuous compounding. Numbers slightly larger than 69.3 are necessary to calculate the doubling time when the compounding is done annually or semiannually. Bankers sometimes call this "The Law of 72."

Table 1 shows doubling times for several rates of steady growth, while Table 2 shows the size of a growing quantity after several periods of growth, where the times are expressed in units in the doubling time.

TABLE 1

Doubling Times for Different Rates of Steady Growth

Percent growth per year	Doubling time in years
Zero	Infinity
0.5	139.0
1.0	69.3
1.5	46.2
2.0	34.7
3.0	23.1
4.0	17.3
5.0	13.9
10.0	6.93
20.0	3.47

TABLE 2

Steady Growth for Different Numbers of Doubling Times

Time, in numbers of doubling times	Size of the growing quantity in multiples of the initial size, N_o
Zero	1
1	2
2	4
3	8
4	16
5	32
⋮	⋮
10	1024
⋮	⋮
n	2^n

This concept of the doubling time is applicable for an increase in size by any factor. The time T_6 for N to increase the size by a factor of 6 is,

$$\begin{aligned} T_6 &= (\ln 6) / k \\ &= 1.79 \ldots / k \\ &= 179 / R \end{aligned}$$

When the quantity N is decreasing in size by a constant fraction per unit time, k is negative, and the doubling time becomes the *half-life*, which is the time for N to decay to half of its initial value, N_o. The decays of radioactive materials are characterized by half-lives.

POWERS OF TWO

The concept of the doubling time allows us to make a convenient reformulation of Eq.3.

(7) $$N = N_o\, 2^{(t/T_2)}$$

This equation is very useful because powers of two are easier to calculate in one's head than powers of e.

EXAMPLE NO.2.

By what factor does a population increase if it grows 5% per year for 30 years?

The doubling time is approximately (70 / 5) = 14 years.
So (t / T_2 = 30 / 14 = 2.14 . . . doubling times.
Then $N = N_o\, 2^{2.14}$ which is between $2^2 = 4$ and $2^3 = 8$.

With a scientific calculator, $N_o\, 2^{2.14} = 4.4 \ldots N_o$. (The keystrokes for this calculation are given in the Appendix)
Thus the factor we are seeking is 4.4 . . .

EXPONENTIAL GROWTH FOR MANY DOUBLING TIMES

An important feature of steady growth is that after long periods of time (many doubling times), the size of the growing quantity becomes enormous.

For mental calculations, it is convenient to remember that

$$2^{10} = 1024 \approx 10^3$$

So the growth in 25 doubling times can be estimated as follows:

$$2^{25} = 2^{10} \times 2^{10} \times 2^{5} \approx 10^{3} \times 10^{3} \times 32 = 3.2 \times 10^{7}$$

Here is another calculational convenience. Steady growth of n% per year for 69.3 years (approximately one human lifetime in the western industrialized nations) results in an overall increase in the size of the growing quantity by a factor of 2^n. For example, If a population has steady growth of 6% per year for 69.3 years, its size will increase by a factor of $2^6 = 64$. Where one school was needed at the start of the period, 64 schools of the same size will be needed at the end of the period!

EXAMPLE NO.3

The large numbers that come as a consequence of many doublings are the basis for mathematical questions such as this. You are to work for 30 days. You have a choice of a salary of $1000 for all this work, or you can have a salary that starts at one cent the first day and grows exponentially, doubling every day for the 30 days. Which method of payment would you prefer?

On the first day your salary, in dollars, is	$2^0 / 100 =$ \$0.01
On the second day	$2^1 / 100 =$ \$0.02
On the third day	$2^2 / 100 =$ \$0.04
On the thirtieth day your salary is	$2^{29} / 100$
which is	\$5,368,709.12

It can be shown that the total salary for the 30 days is $(2^{30} - 1)/100$ which is $10,737,418.23.

EXAMPLE NO.4

Excluding Antarctica, the land area of the earth is 1.24×10^{14} square meters. If the population of the earth in 1992 is 5.5×10^9 people, and if this population continues to grow steadily at 1.7% per year, when would the population density reach one person per square meter on the dry land surface of the earth? We can use Eq.5.

$$t = (1 / 0.017) \ln(1.24 \times 10^{14} / 5.5 \times 10^{9})$$
$$t = 590 \text{ years}$$

Since we know that one person per square meter is an impossible population density, this arithmetic tells us that world population growth will stop in a time short compared to 590 years.

EXAMPLE NO.5

Assume that the world population growth has been steady at 1.7% per year since the time of Adam and Eve. Calculate when they lived. Again we use Eq.5. Now $N = 5.5 \times 10^9$ and $N_o = 2$ (Adam and Eve).

$$t = (1 / 0.017) \ln (5.5 \times 10^9 / 2)$$
$$t = 1279 \text{ years ago, or about 713 A.D.}$$

This establishes that the growth of world population has not been steady; over most of recent history, the growth has been *faster than exponential.* Today's annual growth rate of approximately 1.7% is much larger than the average growth rate over all of human history!

The annual growth rate of world population may have been 1.9% in the 1970s. If this is correct, then the recent period of decline of the population growth rate from 1.9% to 1.7% is a period of growth that is slower than exponential, i.e., the growth rate is declining. Growth that is slower than exponential, and decreasing to $k = 0$ and $R = 0$, is necessary if the earth is to reach zero population growth.

GENERAL CONCLUSIONS ABOUT GROWTH

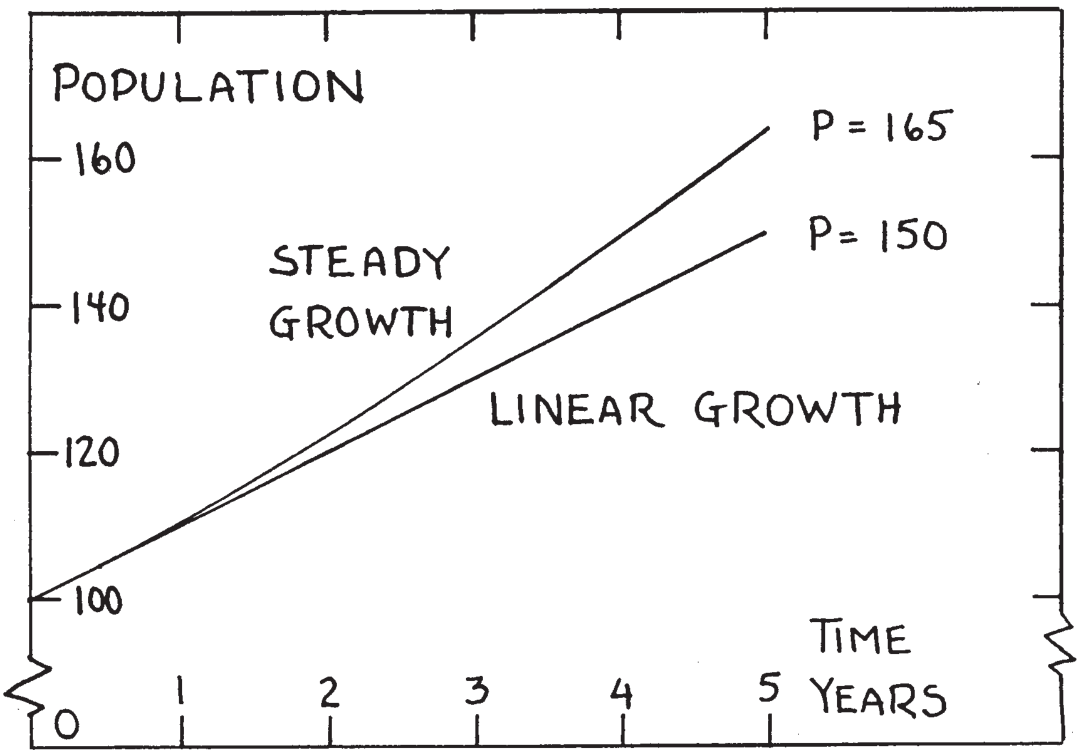
1) Whenever a growing quantity increases by a fixed fraction in a fixed period of time, the growth is steady (exponential).
2) In a modest number of doubling times, the growing quantity will increase enormously in size.
3) As a consequence, the size of things, or the number of things, can never continue to grow indefinitely.
4) In all systems, growth is a short-term transient phenomenon.

The eminent economist Kenneth Boulding summed it all up when he is reported to have said, "Anyone who thinks that steady growth can continue indefinitely is either a madman, or an economist."

The effect of steady growth in the rate of consumption of finite resources such as fossil fuels has been set forth in detail (Bartlett, 1978).

"Sustained Yield" applies to agricultural resources and describes a situation in which the rate of use of a renewable resource equals the rate of biological regeneration through plant growth. "Sustained availability" is a concept that can be applied to nonrenewable resources. For "sustained availability," the rate of consumption of a finite resource must have expo-

FIGURE 2. At time t = 0 we have two populations of 100 persons each. The lower curve represents a linear growth of 10 persons per year. The upper curve represents steady growth of 10% per year. The slope of the curves represent the rate of change of each population. At time t = 0, both populations are changing at a rate of 10 people per year, but because of the steady growth, the rate of change of the upper curve increases with time.



nential decay at a certain rate, and this will allow the resource to be available forever (Bartlett, 1986).

LINEAR GRAPHS

In Fig.2 we see a linear graph of two populations vs. time. Both populations have the value N = 100 people at the time t = 0. The lower curve is the curve of linear growth in which the population increases by 10 people every year. The equation of this line is

$$P = P_o + S\,t$$

where S is the slope, which in this case is S = 10 people per year, and t is the time in years.

$$P = 100 + 10\,t$$

The upper curve is the curve of steady growth in which the population starts at 100 people and increases at a rate of 10% per year.

$$P = 100\,e^{(0.10\,t)}$$

The slope of the curves is the rate of change of the population which initially is the same for both populations. For the lower curve, the slope is constant and has the value of 10 people per year at all times. The upper curve has the same initial slope as the lower curve, but the upper curve gets steeper as time goes on. By differentiating Eq.3, we can find that the slope of the upper curve is,

(8) $dN / dt = K\,N_o\,e^{(k\,t)} = k\,N$ in people per year

Because it deals with differentials, Eq.8 is valid only for times much shorter than the doubling time.

The two curves of Fig.2 are compared in Table 3. We need these two curves and the data of Table 3 to explain an important puzzle.

TABLE 3

Comparison of Linear Growth and Steady (Exponential) Growth

	Linear Growth		Steady Growth	
At end of the year	Population Size People	Rate of change of population People per year	Population Size People	Rate of change of population P/Y at end of yr.
Zero	100	10.0	100	10.0
1	110	10.0	111	11.1
2	120	10.0	122	12.2
3	130	10.0	135	13.5
4	140	10.0	149	14.9
5	150	10.0	165	16.5

A PUZZLE

When the population of 100 people grows steadily at the rate of 10% per year, Table 3 shows that the increase of population in the first year is not 10 people, but is 11 or 11% of the initial population. How can a 10% annual growth rate give an annual increase of 11%?

The answer is that at the end of one year, an annual rate of 10% compounded continuously gives a yield which is the same as an annual rate of 11% compounded once a year. This explains bank ads for savings accounts that give numbers such as these: "Rate, 7.85%; Yield, 8.17%." The ads tell us that $1 left in the bank for one year at a rate of 7.85% compounded continuously (growing exponentially) will, at the end of one year, have the value of $1.0817 because,

$$e^{(0.0785 \times 1)} = 1.0817 \ldots$$

This is the same annual yield as one would have at a rate of 8.17% compounded once a year.

Let us extend this to ask, what is the doubling time if one had steady growth at the rate of 69.3% per year? Eq.6 suggests that T_2 would be one year. How can a rate of 69.3% per year give an increase of 100% in one year? This is a more dramatic example of the puzzle that we just examined and it has the same explanation. *Steady growth at an annual rate of 69.3% gives an annual yield of 100%.*

People sometimes mistakenly suggest that Eq. 6 is approximate, and is valid only for small rates of growth. Equation 6 is exact and correct for all rates of growth.

SEMI-LOGARITHMIC GRAPHS

On a semi-logarithmic graph, a straight line represents steady growth. This is important in recognizing whether or not a series of data points represent steady growth. To see this property, let us take the natural logarithms of both sides of Eq.3.

$$\ln N = (\ln N_o) + k\,t$$

If we plot (ln N) vs. t we have a straight line whose slope is k and whose intercept at time $t = 0$ is ($\ln N_o$).

One uses a scientific calculator to "look up" the values of the log-

TABLE 4

Early Data From the U.S. Census

Year	Population	Nat. Log ln	Log to base 10 log
1790	3,929,214	15.1839...	6.5943...
1800	5,308,483	15.4848...	6.7249...
1810	7,239,881	15.7951...	6.8957...
1820	9,638,453	16.0812...	6.9840...
1830	12,860,702	16.3696...	7.1092...
1840	17,063,353	16.6524...	7.2320...
1850	23,191,876	16.9593...	7.3653...
1860	31,443,321	17.2636...	7.4975...

arithms. Natural logarithms are called up using the "ln" key, and logarithms to the base ten are called up with the "log" key. Either type of logarithm can be used.

EXAMPLE NO.6

Table 4 gives data on the early history of the census in the United States showing, for each decade, the population, the natural logarithm of the population, and the logarithm to the base ten of the population. Two semi-log graphs are shown in Fig.3. Fig.(3a) shows the natural logarithms vs. time from Table IV, and Fig.(3b) shows the logarithms to the base ten vs. time. In both cases the points fall remarkably close to straight lines. The straight line is a visual clue that the population of the United States grew steadily (exponentially) in the period from 1790 to 1860.

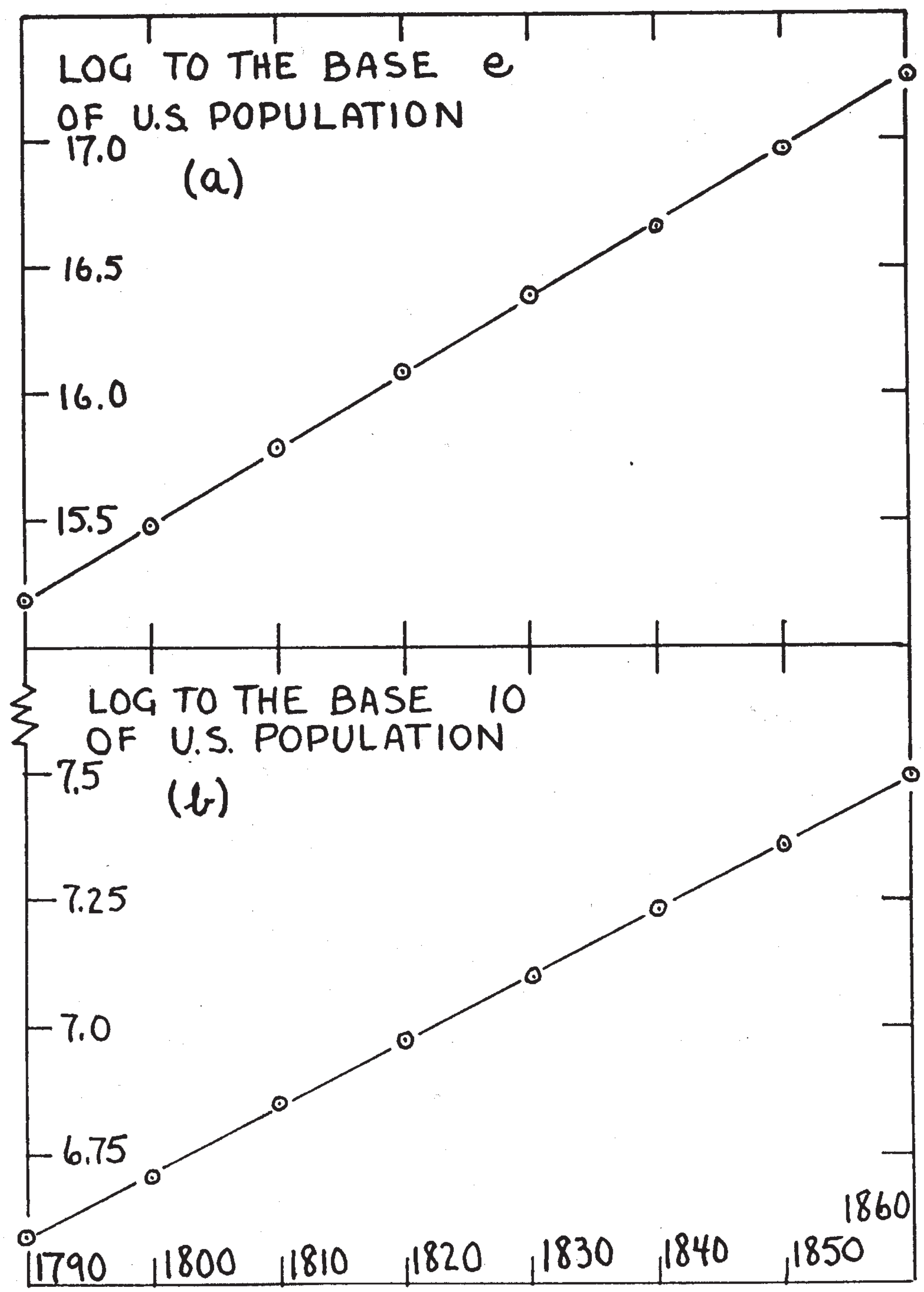
We can calculate the average growth rate in the 70 year interval from 1790 to 1860.

$$k = (1 / 70) \ln (31{,}443{,}321 / 3{,}929{,}214)$$
$$k = (1 / 70) \ln (8.0024 \ldots)$$
$$k = (1 / 70)\ 2.0797 \ldots$$
$$k = 0.0297 \ldots \text{ or } R = 2.97\% \text{ per year}$$

From Eq.(6), the doubling time, $T_2 = 23.3$ years.

The numbers of this example are particularly convenient for a "powers of two" calculation, which one can do in one's head. The 1860 population

FIGURE 3. Linear graphs of the logarithms of the population of the United States from 1790 to 1860. The upper curve (a) shows natural logarithms and the lower curve shows logarithms to the base ten. In each graph the data points fall very close to a straight line, which is visual evidence that the population growth of the United States in this period was steady (exponential).



is very nearly eight times the 1790 population, (8.0024). Eight is three doublings, which took place in 70 years. So T_2 = 70 years / 3 = 23 years.

SEMI-LOGARITHMIC GRAPH PAPER

The most convenient way to make a semi-log plot is to use semi-log graph paper. This paper is printed in such a way that distances on the vertical scale are proportional to the logarithms of the quantity being plotted, while the horizontal scale is the usual linear scale. This saves one the need to look up logarithms.

Important properties of semi-log paper are that all of the decades of change on the vertical scale are represented by the same distance, and there is no zero on the vertical scale. Thus the distances on the vertical scale are the same for the numbers from 1 to 10, as from 10 to 100, as from 100 to 1000, etc. Semi-log graph paper can be purchased at bookstores that have engineering supplies. In buying semi-log paper, one must specify how many cycles one wants. "One-cycle" paper will accomodate one decade of data; "two-cycle" paper will handle two decades of data, as for example from 1000 to 100,000. "Five-cycle" paper will accommodate data ranging over five decades.

In the example of United States population from 1790 to 1860, the population numbers range from 3.9 million to 12.8 million. This will require two-cycle paper, one cycle for the range 1 to 10 million, and the second cycle for the range from 10 to 100 million. This graph is shown in Fig.4. The data points are surrounded by circles which are bracketed by error bars. In this case the vertical error bars show an arbitrary range of plus or minus 5%. The point in showing these arbitrary error bars is to show that on a semi-log graph a fixed fractional error, such as 5% is represented by a constant vertical distance, no matter where one is on the curve.

DOUBLING TIMES FROM SEMI-LOG GRAPH PAPER

The graph on semi-log paper allows interpolations and extrapolations. From Fig.4 one can read that the U.S. population reached 8.0 million at the start of the year 1814 and reached the population of 20.0 million about the date 1845.2.

The doubling time for the early growth of the U.S. population can be read directly from a graph on semi-log graph paper, without the need to do

FIGURE 4. Semi-log graph of the population of the United States for the period 1790 to 1860, plotted on two cycle semi-log graph paper. Notice that the distance on the vertical axis from 1 million to 10 million is the same as the distance from 10 million to 100 million. As in Fig.3, the points fall nicely on a straight line. The vertical error bars illustrate the size of an hypothetical uncertainty of plus or minus 5%.

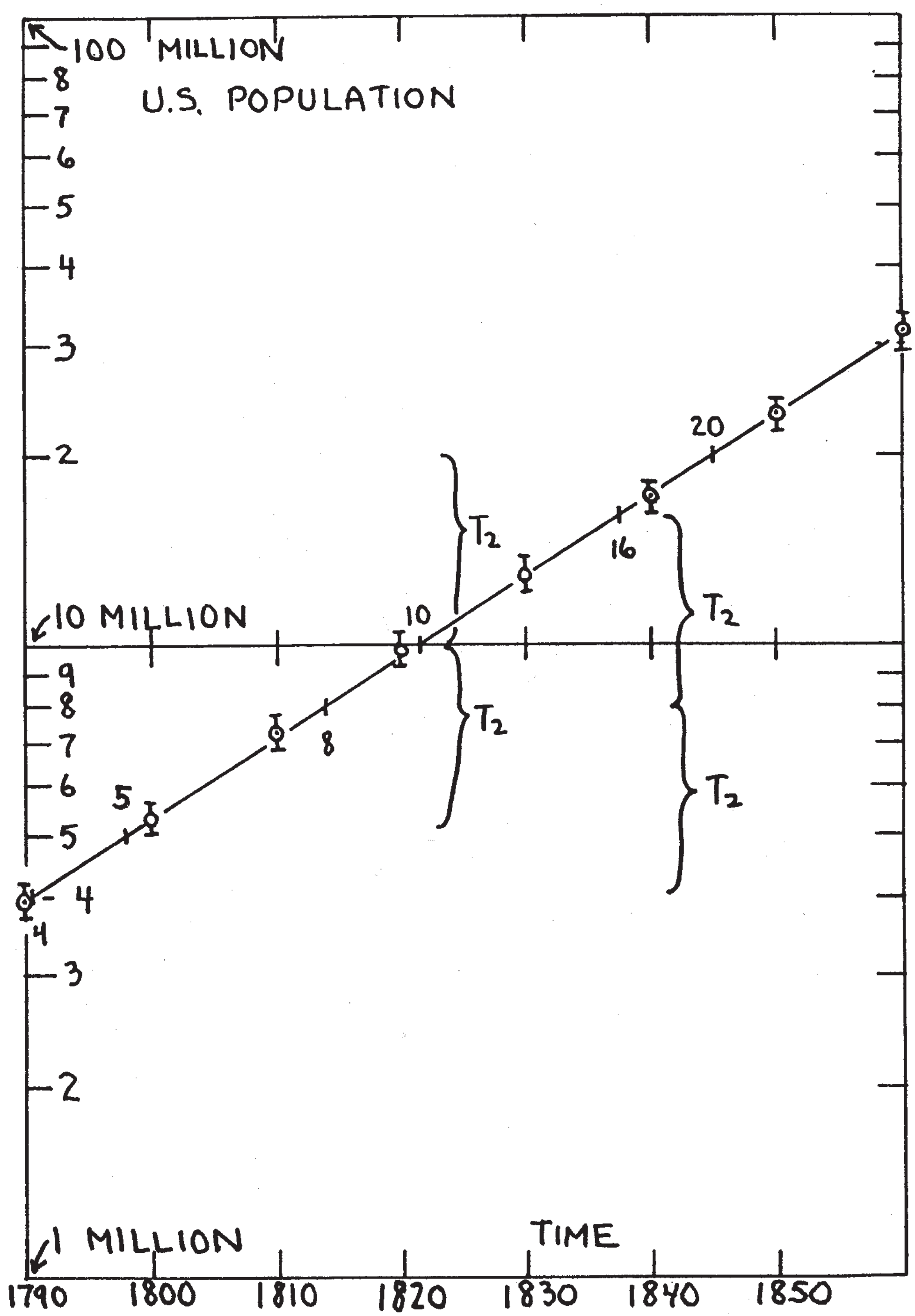


TABLE 5

Times for a Succession of Doublings as Read From the Graph of Fig.(4)

Population, in millions	Date, years
First Series of Points	
5.0	1798.0
10.0	1821.6
20.0	1845.2
Second Series of Points	
4.0	1790.5
8.0	1814.0
16.0	1837.6

exponential arithmetic. We need to read the times for a succession of doublings. In Fig.4, we can read two sets of points as shown in Table 5. In the first set of points, two doublings were observed in the 47.2 years between 1798.0 and 1845.2. Thus one doubling time is (47.2 / 2) = 23.6 years. In the second set of points, two doublings were observed in the 47.1 years between 1790.5 and 1837.6. Thus one doubling time is (47.1 / 2) = 23.55 years. The difference between these two results is indicative of the precision of this graphical method.

GROWTH FASTER OR SLOWER THAN EXPONENTIAL

Table 6 contains a set of data on world population over a period of about 400 years (World Almanac, 1992). Figure 5 is a semi-log plot of these data. From an examination of Fig.5 we can see that from 1650 to about 1975 the line is curving more steeply upward as time goes on. This indicates growth which is *faster than exponential*. Sometime after 1975 and for the projections to the year 2025, the curve is becoming less steep, which indicates growth that is *slower than exponential*.

The third column in Table 6 tabulates the average annual growth rates for each of the intervals in the data, as calculated from Eq.4, while the fourth column lists the dates of the mid-points of each of the intervals in the data.

Figure 6 shows a plot of the third and fourth columns of Table 6. Here we see how the value of R , the rate of growth of world population, has

TABLE 6

An Estimate of the Recent History of World Population Growth (8) with a Projection to the Year 2025

Year	Population in billions	Average rate pop. growth % per year	Mid date of period
1650	0.550		
		0.276	1700
1750	0.725		
		0.483	1800
1850	1.175		
		0.617	1900
1900	1.60		
		0.744	1915
1930	2.00		
		1.23	1940
1950	2.56		
		1.78	1967.5
1975	4.00		
		2.27	1977.5
1980	4.48		
		1.83	1983
1986	5.00		
		1.60	1988
1990	5.33		
		1.27	1992.5
1995	5.68		
		1.52	1997.5
2000	6.13		
		1.15	2012.5
2025	8.18		

changed, as indicated by these data. They indicate that the quantity R increased to a maximum value of approximately 2% per year around the year 1975, and then began to decline.

No claim is made that these data are definitive. They are used only as an illustration. There are large uncertainties in the data on world population. It will be good news indeed if further studies confirm that R peaked and that world population growth has made the transition from growth that is faster than exponential to slower than exponential. However, this good news should not blind us to the fact that zero population growth will not occur until R = 0.

FIGURE 5. Semi-log graph of world population from 1650 with projections to 2025. From 1650 to about 1975 the line is curving upward, which represents growth faster than exponential. After about 1985, the curve starts becoming less steep, which represents growth slower than exponential. The curve has an inflection point somewhere between 1975 and 1985. The point at 1930 is shown with an error bar of plus or minus 10%, solely to illustrate the size of an uncertainty of this magnitude.

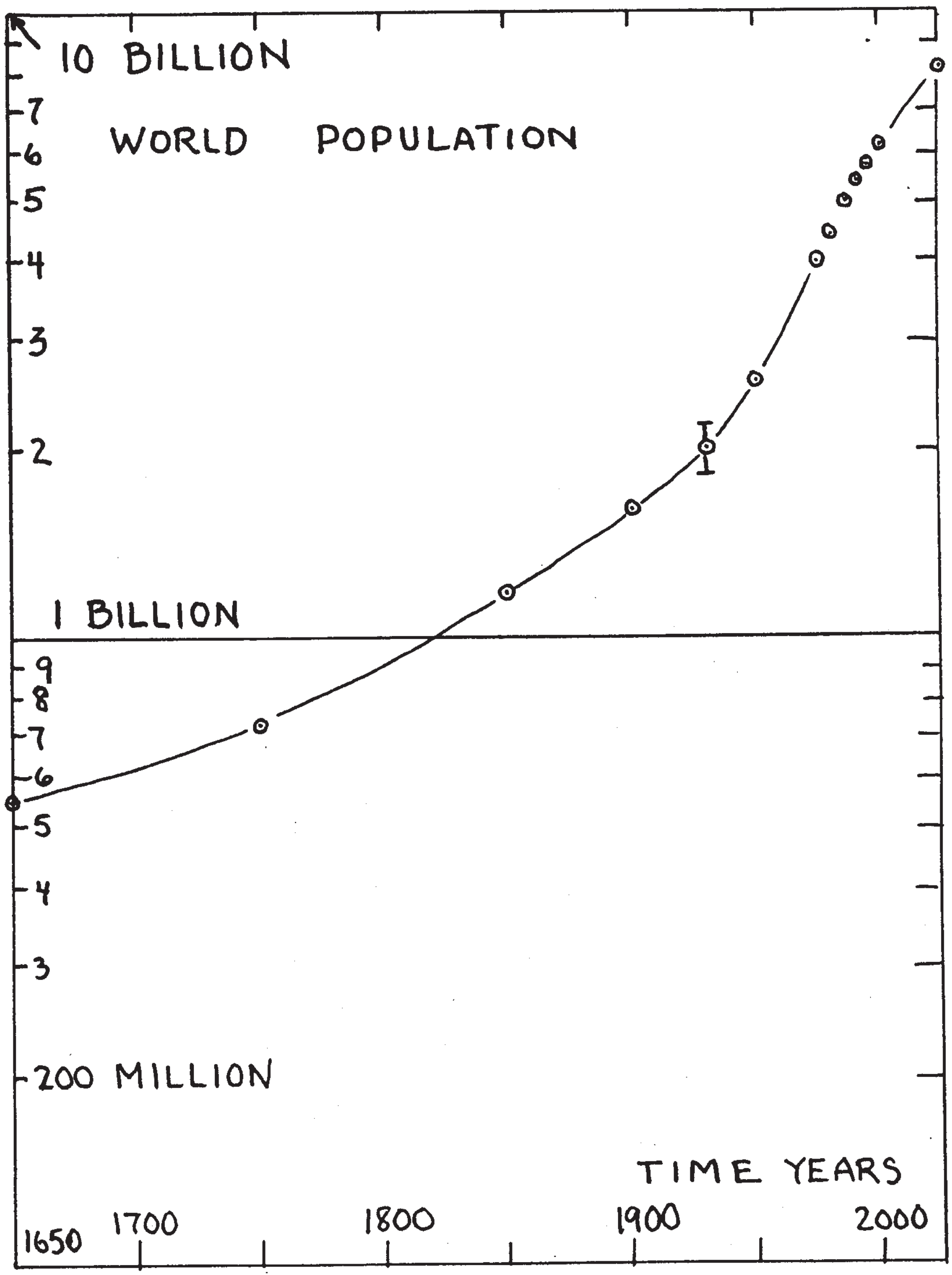
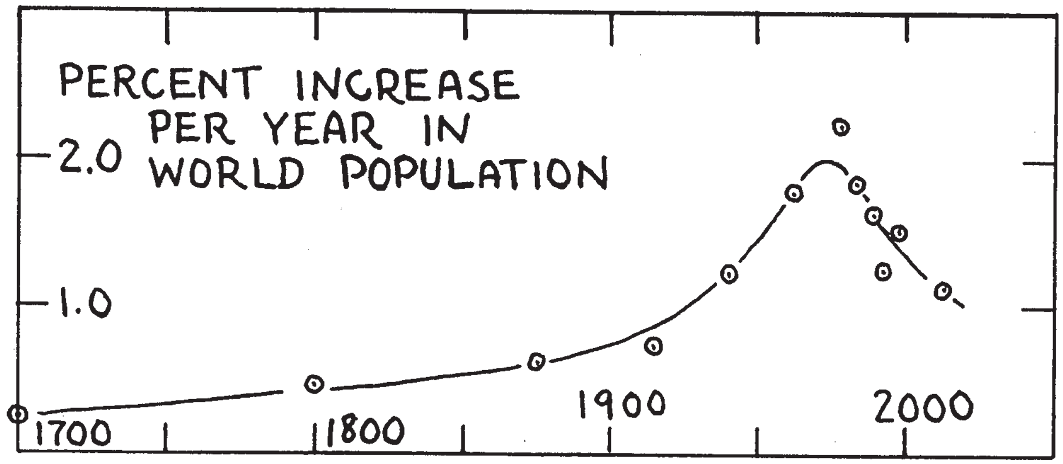


FIGURE 6. The rate of growth R of world population in percent per year as a function of time, as calculated from the data shown in Fig.5. The points scatter considerably, but they suggest a peak growth rate of about 2% per year was reached around 1975. The uncertainties in these rates of growth could be as large as plus or minus a half percent.



The facts of population growth are often misrepresented.

"Human population stayed almost constant at nearly 500 million from the year 0 to 1500. Then it began rising exponentially. It doubled between 1850 and 1950, doubled again between 1950 and 1990 . . ." (Newsweek, 1992). The fact that the second quoted doubling time is shorter than the first indicates growth that is *faster than exponential.* It would be more accurate to say that the world population probably grew very slowly from the dawn of time up until a few hundred years ago. In the last few centuries the growth rate has increased rapidly which has resulted in population growth that has been faster than exponential. The growth rate may have peaked around 1975 with the consequence that the growth since 1975 has been slower than exponential.

"The United States is close to ZPG with a fertility rate of 2.1 . . ." (Popular Science, 1992). The U.S. Census figures for 1980 and 1990 show that the average growth rate was 0.933% which translates to a net increase of 2.3 million people each year. The alleged undercount, and illegal immigration could push this annual increase to 3 million a year. The riots in our large cities underline the fact that the United States is not now able to take proper care of the present U.S. population, and they emphasize the urgency of reducing the U.S. population growth rate to zero and then to negative values as rapidly as possible.

A SERIES OF EXAMPLES

EXAMPLE NO.7

In his last night as host of the "Tonight" show, Johnny Carson observed that when the show started on October 1, 1962 the world population was 3.1 billion, and as he was signing off on May 22, 1992, he said the world population was 5.5 billion. What average growth rate of world population is indicated by these data?
The show ran for 29.64 years, so the value of k is

$$k = (1 / 29.64) \ln (5.5 / 3.1) = 0.0193 \ldots \text{ or } R = 1.93\% \text{ per year}$$

Carson's figures indicate a growth rate somewhat higher than the 1.7% that is quoted for 1992.

What was the world population increase in people per day at the start and end of the run of Johnny Carson's show as indicated by the data Carson cited? For this, we need to know the value of k in units of *per day*.

$$k = 0.0193 / 365 = 0.0000530 \ldots \text{ per day}$$

From Eq.8, the rate of change of a population is k N . This gives us the following rates of change

Oct, 1962; $0.0000530 \times 3.1 \times 10^9 = 1.64 \times 10^5$ people per day
May, 1992; $0.0000530 \times 5.5 \times 10^9 = 2.91 \times 10^5$ people per day

If we want to know the hourly rate of change, we must have the value of k in units of *per hour*.

$$k = 0.0000530 / 24 = 2.208 \times 10^{-6} \text{ per hour}$$

Oct. 1962; $2.208 \times 10^{-6} \times 3.1 \times 10^9 = 6.84 \times 10^3$ people per hour
May, 1992; $2.208 \times 10^{-6} \times 5.5 \times 10^9 = 12.1 \times 10^3$ people per hour

These numbers show the dramatic changes that have taken place in the period that one person hosted a popular television show.

EXAMPLE NO.8

It is reported that "North American waterfowl populations have declined 30% since 1969, mostly due to the loss of wetland habitat" (Clearing House Bulletin, 1992).

What is the average annual rate of loss which would reduce the waterfowl population to 70% of its original population in 22 years ? From Eq.4

$$K = (1 / 22) \ln (0.70) = (1 / 22) (-0.356 \ldots) = -0.0162 \ldots$$

The numbers indicate the average rate of habitat loss over the 22 years was 1.62% per year.

A loss rate of 1.62% per year is so small as to seem to be trivial. Yet it adds up to a very significant loss in a modest number of years.

If this rate of loss continued for 50 years from 1969 to 2019, what fraction of the waterfowl would remain ?

$$(N / N_o) = e^{(-0.0162 \times 50)} = e^{(-0.81)} = 0.44 \ldots$$

Only about 44% of the population would remain.

·EXAMPLE NO.9

It is reported that "World food production will have to increase threefold in the next 40 years to meet the needs of an estimated nine billion people" (Gasser & Fraley, 1992).

What is the average annual rate of increase of food production needed to meet this goal ? Tripling implies that $(N / N_o) = 3$. Thus, from Eq.4

$$k = (1 / 40) \ln 3 = 0.0275 \text{ or } R = 2.75\% \text{ per year}$$

EXAMPLE NO.10

"The population of fishermen in Colorado has experienced a fourfold increase in the last 35 years" (Engle, 1992).

What is the average rate of growth of fisherman in Colorado in this period ? From Eq.4,

$$k = (1 / 35) \ln 4 = 0.0396 \text{ or } R = 3.96\% \text{ per year}$$
$$T_2 = 69.3 / 3.96 = 17.5 \text{ years}$$

Notice that the numbers in this example allow us to work this in our head. The "fourfold increase" is exactly two doublings. Two doublings in 35 years means one doubling in half of this, or 17.5 years. The rate of growth can be found by dividing 70 / 17.5 = 4% per year.

EXAMPLE NO.11

In 1983 the ski area of Vail, Colorado celebrated its 20th anniversary, and for a short period all-day lift tickets were sold at the 1963 price of \$5 instead of at the 1983 price of \$20. Calculate the average rate of inflation of the cost of these ski lift tickets, and predict the cost of ski lift tickets at Vail if this inflation rate continues to 1993 and to 2003.

The numbers in this example allow easy mental calculation. The cost of lift tickets increased by a factor of four (two doublings) in 20 years. The doubling time is then 10 years. Then, from Eq.6, 70 / 10 = 7% is the average annual rate of inflation of lift tickets. If this rate continues, lift tickets will double in cost every decade; the cost in 1993 will be \$40 and in 2003 it will be \$80 . . . The 1992 cost of lift tickets may already have exceeded \$40.

EXAMPLE NO.12

"The population of the three former French colonies of North Africa, Tunisia, Algeria, and Morocco, has nearly tripled over the past three decades, . . ." (Randal, 1992).

What is the average rate of growth that gives tripling in 30 years ? From Eq.4,

$$k = (1 / 30) \ln (3) = 0.0366 \text{ or } R = 3.66\% \text{ per year}$$

EXAMPLE NO.13

It was noted recently that there are 760,000 lawyers in the United States in 1992, and this number is increasing at an annual rate of 3.64%. It was assumed that the annual rate of increase of the population of the United States is "0.6% (a typical figure)," and it was then calculated that if these rates continued, lawyers would be half of the United States population by the year 2188 (Seligman, 1992).

The average annual growth rate of the United States population in the 1980s was 0.933%, rather than 0.6%. Let us do the calculation using 0.933%. Since the number of lawyers is for 1992, we must estimate the United States population in 1992.

$$\begin{aligned} N_{1992} &= 248.71 \times 10^6 \, e^{0.00933 \times 2} \\ &= 253.40 \times 10^6 \text{ people} \end{aligned}$$

If we assume that the growth rates do not change, the equation for the population of the United States after 1992 is

$$N_p = 253.40 \times 10^6 e^{0.00933 t}$$

and the equation for the number of lawyers in the United States is

$$N_L = 7.60 \times 10^5 e^{0.0364 t}$$

The question asks, when will lawyers be half of the population, or when is

$$N_L = N_p / 2?$$

$$7.60 \times 10^5 e^{0.0364 t} = (253.40 \times 10^6 / 2) e^{0.00933 t}$$

This equation must be solved for t.

$$e^{(0.0364 - 0.00933) t} = 253.40 \times 10^6 / (2 \times 7.60 \times 10^5)$$

$$e^{0.0271 t} = 1.667 \times 10^2$$

$$0.0271 t = \ln(1.667 \times 10^2)$$
$$t = 189 \text{ years}$$

In that year the population of the United States would be

$$N_p = 253.40 \times 10^6 e^{0.00933 \times 189}$$

which is approximately 1.5 billion people, half of whom would be lawyers.

In this example, the population of lawyers is growing faster than the general population, with the differences in the growth rates being described by $k = 0.0271$ per year. This k has a doubling time (Eq.6) of about 25 years. So after 25 years past the year 2181, the entire United States population would be lawyers!

These calculations are a nice example of *reductio ad absurdum*. It was proposed that the growth rates of the population of people and of lawyers would remain constant for a long period of time. This led to the conclusion that the United States population would be 50% lawyers in 189 years. The absurdity of the conclusion proves the absurdity of the assumption that the growth rates can remain constant for long periods of time. *Thus the term "sustainable growth" is an oxymoron!*

CONCLUSION

It is hoped that this tutorial, with its worked out numerical examples, will assist readers in gaining a better understanding of the nature of growth, and an improved facility in interpreting data on population growth.

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APPENDIX

This appendix contains the formula for the average of a quantity that is growing exponentially, and the key strokes needed to work a few of the examples on a "scientific calculator".

It can be shown that when N grows in accord with Eq.3, the average value of N in the interval from t = 0 to t = T is

$$N_{av} = (1 / T) \int_0^T e^{kt}\, dt = (N_o / k\, T)\, [e^{kT} - 1]$$

EXAMPLE NO. 1

KEY	DISPLAY READS
On	0
248.61	248.61
Division sign	248.61
226.55	226.55
Equals sign	1.09737...
ln	0.0929197...