Arithmetic of Growth: Methods of Calculation, II

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This calculational tutorial continues the presentation of an earlier article (Bartlett, 1993). It starts with a news item that features one elderly person who has 67 grandchildren and 201 great grandchildren. This tutorial develops simple mathematical models to show how to calculate approximate average rates of growth of descendants using very simple assumptions plus the data from the news story. The model is then enlarged to describe the growth of populations and to see how the growth of populations is related to the growth of descendants and to fertility. The analysis is then generalized so it can be applied to other reproductive phenomena, such as the production of Ph.D.s. The goal is to illustrate the essential features of the simplest elements of the population growth process by introducing modeling that is within the reach of those who can use algebra.

INTRODUCTION

A news story told of a 98-year old person who:

> will never lack for heirs; the infant [pictured] is [the] *200th* great grandchild. That total now stands at 201, and with 67 grandchildren in the clan, [the person] probably isn't finished counting" (*LIFE,* 1993).

This story suggests quantitative questions such as:

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1) a) What is the average number of children per woman (fertility) in each generation, b) averaged over several generations, based on the number of grandchildren or on the number of great grandchildren?

2) What is the average annual growth rate of the number of this person's descendants from one generation to the next, and averaged over several generations?

3) If these rates of increase continue unchanged, how many great great grandchildren, etc. can be expected?

4) What is the average growth rate of a population that experiences these rates of production of offspring?

5) If a couple has a given number of descendants in the nth generation, what is the population growth rate that would follow if all couples have this same number of descendants in that generation?

REVIEW

The important arithmetic of steady growth has been presented in earlier papers, (Bartlett, 1978; 1993). Following the analysis of these earlier papers, let S_o be the size of a growing quantity at a starting time, and let S be the size of the quantity after the passage of a time interval t , expressed in years. The *average annual fractional rate of growth* (FR), k , is given in Eq. 4, Pg. 365 of Bartlett (1993).

$$FR = k = (1 / t) \ln (S / S_o) \qquad (1)$$

The quantity $\ln (S / S_o)$ is the natural logarithm of the quotient (S / S_o). The doubling time for this growth is $T_2 = (\ln 2) / k$. To find the *average annual percent rate of growth* (PR) , one multiplies k by 100.

Example 1: "It has taken twelve years to quadruple the national debt . . ." (Skaggs, 1993).

Calculation 1: From these figures, calculate the PR of the national debt.

The present debt S is said to be four times the debt S_o twelve years ago, so $(S / S_o) = 4$ and $t = 12$ y . From Eq. 1 the FR is,

$$k = (1 / 12) \ln (4) = 0.116 \text{ per year}$$

This corresponds to an PR of 11.6 % .

Calculation 2: At this rate, how long would it take for the national debt to increase by a factor of ten?

If we solve Eq. 1 for t we have:

$$t = (1/k)\ln(S/S_o) = (1/0.116)\ln(10/1)$$

$$t = 8.66\ln(10) = 8.66 \times 2.303 = 19.9\ y$$

At this PR , the national debt would increase by a factor of approximately ten in two decades!

Calculation 3: If the PR of the U.S. population continues at its 1996 rate of approximately 1.0 % per year, by how much will the *per capita* national debt increase in 19.9 years?

For an PR of 1.0 % per year, k = 0.01 per year. The population growth is therefore described by:

$$P_{19.9} = P_o\, e^{(0.01 \times 19.9)} = 1.22\, P_o$$

where e is the base of natural logarithms: e = 2.718 . . . This result tells us that in 19.9 years (2013) the population would be about 22 % greater than in 1993, while the debt would be a factor ten higher. Thus the *per capita* national debt would be 10.0 / 1.22 = 8.2 times as high as in 1993.

PARENT AND DESCENDANTS

We will consider a population system that consists of a single ancestor and the ancestor's direct descendants. To start the investigation we need definitions, and a model of the system.

Let us use the term "cohort" to refer to all members of a group of individuals who bear a common relationship to one ancestor. We will designate the ancestor as the "parent" and as the zeroth cohort; all the parent's children constitute the first cohort; all the grandchildren constitute the second cohort, etc. Although it is wildly unrealistic, we will make the assumption that all members of a given cohort are born simultaneously. We know that this is not true, so this will limit the accuracy of calculations made using this model. However, the model is still of value because it gives reasonable approximate results, and it lets us explore the numerical dynamics of the system.

Let us use t_g to represent the generation time in years. This is the time from the birth of one cohort to the time of birth of the following cohort. Typically t_g is of the order of 20 to 30 years. These designations are illustrated in Table 1.

TABLE 1

This table illustrates the time scale in generations, assuming the generation time is 30 years. The table shows the relation of the time scales to the number of the cohort, and shows the common name of the cohort as it relates to the parent that constitutes the zeroth cohort. The time is the time at the start of the lives of the members of the numbered cohort.

Cohort #	Time (Years)	Cohort Number	Name
0	t = 0	Zeroth	Parent
1	t = 30	First	Children
2	t = 60	Second	Grandchildren
3	t = 90	Third	Great Grandchildren
n	t = 30 n	nth	(Great)$^{(n-2)}$ Grandchildren

SIMPLE MODEL

We want to derive the expression for the FR of the growth of the number of descendants of one parent. We will not be counting spouses. In order to calculate the k_d of Eq. 1 (the subscript d is for descendants) we need to know the number of descendants of one parent at the start and end of a known time interval.

We will start at time t = 0 with the birth of one person who will be the parent and who is the zeroth cohort. At time t_g , the parent has N children (the first cohort). The size C_1 of the first cohort is N . We will assume that N is a constant through all the generations, and that all offspring live to maturity and reproduce. At a time 2 t_g each of these N children has N offspring who are grandchildren (the second cohort). The size C_2 of the second cohort is $C_2 = N \times N = N^2$ people. At the time 3 t_g each of these N^2 grandchildren has N offspring who are great grandchildren (third cohort). The size of the third cohort is $C_3 = N^3$ people. The size of the nth cohort will be:

$$C_n = N^n \tag{2}$$

From Eq. 2, the ratio of the size of a given cohort to the size of the preceding cohort is:

$$C_n / C_{(n-1)} = N \tag{3}$$

The interval between the births of these two cohorts is t_g . We can use the ratio of Eq. 3 in Eq. 1 to find an expression for the average annual FR in the number of descendants of the one parent:

$$k_d = (1 / t_g) \ln (N) \tag{4}$$

There is some convenience in expressing k_d in terms of the size C_n of the nth cohort. From Eq. 2:

$$N = C_n^{(1 / n)}$$

so that:

$$k_d = (1 / t_g) \ln (C_n^{(1 / n)})$$

or:

$$k_d = (1 / t_g) (1 / n) \ln (C_n) \tag{5}$$

We recognize that the quantity $n\ t_g$ in the denominator of the right side of Eq. 5 is the total time from the birth of the parent to the birth of the nth cohort. Let us define this time to be T . [In practice, T would be the number of years from the birth of the parent to the mean date of birth of the members of the nth cohort.] Equation 5 for k_d can now be written as:

$$k_d = (1 / T) \ln (C_n) \tag{6}$$

Equations 4, 5, and 6 are convenient alternative expressions for the value of k_d . To use Eq. 4, one must know values for t_g , and N ; to use Eq. 5, one must know values for t_g , n , and C_n; and to use Eq. 6, one must know values for T and C_n .

Example 2: Data: Suppose we have a system where N = 3 descendants per person, the generation time t_g is 30 years, and we consider the fourth cohort (n = 4) (great great grandchildren) whose mean date of birth is T = 120 years after the birth of the parent.

From Eq. 2, the population of the fourth cohort is $C_4 = 3^4 = 81$ people.

Calculation 1: Use Eq. 4 to find the the FR of the descendants through the fourth cohort.

$$k_d = (1/30) \ln(3) = 0.037 \text{ or } 3.7\ \% \text{ per year}$$

Calculation 2: Use Eq. 5 to find the FR of the descendants through the fourth cohort.

$$k_d = (1/30)(1/4) \ln(81) = 0.037 \text{ or } 3.7\ \% \text{ per year}$$

Calculation 3: Use Eq. 6 to find the FR of the descendants through the fourth cohort. In this case T = 4 x 30 = 120 years.

$$k_d = (1/120) \ln(81) = 0.037 \text{ or } 3.7\ \% \text{ per year}$$

Equations 4, 5, and 6 use different related data from one problem but, as is shown in the three calculations above, they all give the same result.

A MORE REALISTIC MODEL

It is not realistic to assume that all members of a given cohort have the same number of N of offspring, nor is it realistic to assume that the average number of offspring N per person remains constant from one cohort to the next. So we now repeat the analysis of the previous section, taking account of the fact that we can have different numbers of offspring per person in the different cohorts and that, within one cohort, not all individuals will have the same number of offspring; some may not live to have offspring, and some may live but have no offspring.

Again, we will let C_n be the total number of persons born in the nth cohort. We will start with the parent, who we will label as the zeroth cohort. The size of the zeroth cohort is $C_o = 1$. That person has N_1 children who are the first cohort, ($C_1 = N_1$) . These children produce a total of C_2 offspring, (grandchildren) who are the second cohort. Let N_2 be the average number (*arithmetic* mean) of grandchildren born in the second cohort per child born in the first cohort:

$$N_2 = C_2 / C_1 = C_2 / N_1$$

so that:

$$C_2 = N_1 N_2.$$

Note: we are counting all of the N_1 children that were born in the first cohort, whether or not they later have offspring. (In the Simple Model, N was the number of offspring per person, which assumed that all offspring lived and produced offspring.) These grandchildren have a total of C_3 offspring who are great grandchildren and who are the third cohort. Let N_3 be the average (*arithmetic* mean) number of grandchildren born in the third cohort per child born in the second cohort:

$$N_3 = C_3 / C_2 = C_3 / N_1 N_2 \qquad (7)$$

As a result:

$$C_3 = C_2 N_3 = N_1 N_2 N_3$$

In general, for the nth cohort:

$$C_n = C_{(n-1)} N_n = N_1 N_2 N_3 \ldots N_n \qquad (8)$$

Our goal is to use data from n cohorts to find, on the average, the ratio of the size of one cohort to the size of the preceding cohort so that we can insert that ratio in Eq. 1 to find k_d , the FR of the number of descendants.

The Ns are generally all different, so we will let Eq. 8 define an average N , to be called N_{an} , such that each of the different Ns on the right side of Eq. 8 can be replaced with N_{an} . The two subscripts refer to the average (a) based on data through the cohort numbered (n) . The quantity N_{an} is the average number of persons born in one cohort per person born in the preceding cohort. Equation 8 defines N_{an} as is shown in Eq. 9:

$$C_n = N_1 N_2 N_3 \ldots N_n = (N_{an})^n \qquad (9)$$

If we raise each of the terms of Eq. 9 to the power (1 / n) we have:

$$N_{an} = (N_1 N_2 N_3 \ldots N_n)^{(1/n)} = (C_n)^{(1/n)} \qquad (10)$$

The left equality of Eq. 10 defines N_{an} to be the *geometric* mean of the Ns . The *geometric* mean of a set of numbers will generally have a different value than the more conventional *arithmetic* mean of the same set of numbers. One mean cannot be substituted for the other. The right term gives us an expression for N_{an} in terms of the size C_n of the nth cohort.

Equation 9 shows that the size C_n of the nth cohort is N_{an}^n. From this it follows that the desired ratio of the size of the nth cohort divided by the size of the (n − 1) cohort is N_{an} :

$$C_n / C_{(n-1)} = N_{an}$$

We can use Eq. 1 now to find the FR of the number of descendants:

$$k_d = (1 / t_g) \ln (N_{an}) \qquad (11)$$

The data over several cohorts are used to calculate the value of N_{an} . We can use Eq. 10 to express N_{an} in terms of the more accessible C_n :

$$k_d = (1 / t_g) \ln (C_n^{(1 / n)})$$

This can be rearranged to give:

$$k_d = (1 / t_g) (1 / n) \ln (C_n) \qquad (12)$$

This can be expressed in terms of the time T :

$$k_d = (1 / T) \ln (C_n) \qquad (13)$$

Equation 11 is closely parallel to Eq. 4: the only difference is that the constant N of Eq. 4 has been replaced by the *geometric* average N_{an} in Eq. 11. Equation 12 is identical to Eq. 5 and Eq. 13 is identical to Eq. 6.

The increase in the number of descendants in the succession of cohorts is illustrated in Table 2.

Example 3: Show the difference between the *geometric* mean and the *arithmetic* mean in finding the value of N_{an} .

Calculation 1: If there are 5 children born in the first cohort ($N_1 = C_1 = 5$); and if 15 grandchildren are born ($C_2 = 15$), then the average number of grandchildren per child is, from Eq. 3, $N_2 = 15 / 5 = 3$. The *arithmetic* mean of 5 children born in the first cohort per person in the zeroth cohort, and 3 in the second cohort for each person born in the first cohort, is (5 + 3) / 2 = 4 children per person.

Let's test this *arithmetic* mean to see if 4 children per person will give 15 grandchildren. If each of 4 persons in the first cohort (the children) has 4 children, the size of the second cohort (the grandchildren) will be 16 and not the 15 of this example. The *arithmetic* mean does not give us the desired average number of offspring per person. The *geometric* mean of 5 and 3 is:

$$N_{a2} = (5 \times 3)^{(1 / 2)} = (15)^{(1 / 2)} = 3.87 \text{ children per person}$$

So, if one person has 3.87 children (the first cohort), and these each have an average of 3.87 children (the second cohort), then the size of the

TABLE 2

Sizes of successive cohorts of descendants of a single individual. The Ds are the sizes of the cohorts, and D_n is the number of persons born in the nth cohort. The Ns are the average number of children born in one cohort per person born in the preceding cohort. We start counting with the one person who is the zeroth cohort. Spouses are not counted here.

Generation	Direct Descendants	
0	$C_o = 1$	
1	$C_1 = N_1$	Children
2	$C_2 = N_1 N_2$	Grandchildren
3	$C_3 = N_1 N_2 N_3$	Great Grandchildren
n	$C_n = N_1 N_2 \ldots N_n$	(Great)$^{(n-2)}$ Grandchildren

For the case where $N_1 = N_2 = N_3 = \ldots N_{an}$, i.e., the number of children per person is the same for all cohorts and is equal to the average N_{an} :

0	$C_o = 1$	
1	$C_1 = N_{an}$	Children
2	$C_2 = N_{an}^2$	Grandchildren
3	$C_3 = N_{an}^3$	Great Grandchildren
n	$C_n = N_{an}^n$	(Great)$^{(n-2)}$ Grandchildren

second cohort (the grandchildren) will be (3.87) × (3.87) = 15 people, which is what was observed.

Example 4: The 4th cohort consists of 405 individuals (great great grand-children).
Calculation 1: Find the (*geometric*) mean number of offspring per person, averaged over the four generations. From the outside terms of Eq. 10:

$$N_{a4} = (405)^{(1 / 4)} = (405)^{(0.25)} = 4.49$$

This tells us that if, in each of the four cohorts, each individual born in one cohort produces an average of 4.49 offspring, there will be 405 great great grandchildren.

NUMBER OF ANCESTORS

Many years ago, after I had lectured on this arithmetic and the way it applies to the growth of populations, an anonymous note appeared in my mailbox with a tongue-in-cheek message along these lines.

> Professor Bartlett, you are all wrong about a population explosion! Quite the contrary; the population is *declining* rapidly. Note that I have two parents (2^1), four grand parents (2^2), eight great grandparents (2^3), and for the nth generation back I have (2^n) (great $)^{(n-2)}$ grandparents. Go back far enough and the population was once infinitely large, but now it comes down to just me. The human race is vanishing rapidly!

GROWTH OF POPULATIONS

It is important to note that the FR of the descendants of one person is not the same as the FR of a population. The reason is that "It takes two to tango." In the previous derivations we attributed all of the descendants to one parent; none were attributed to the parent's spouse.

We now want to use these concepts to model the growth of populations. For simplicity, we will confine our attention to long-term stable monogamy. Every person needs a spouse in order to produce offspring, and the spouses come from other families, so each couple's children are counted as the descendants of two family lines. It follows then that in a regime of a fixed constant fertility, the FR of population will be less than the FR of the descendants of one parent.

So, in order to calculate population growth rates, we must make a population model.

A POPULATION MODEL

Again our goal is to create a population model so that we can determine the ratio of the population of the nth cohort divided by the population of the (n − 1) cohort. This will then be inserted into Eq. 1 in order to find the value of the FR of the population. In order to see the dynamics of the population growth, let us model an isolated growing community.

We will let c_n be the total number of children born in this community as part of the nth cohort. Lower case "c" is used for population cohorts which are different from cohorts of descendants for which capital "C" is used. At the time $t = 0$, we will start with the birth of c_o individuals (half male and half female) who will constitute the zeroth cohort. Let the length of the generation time be $t_g = 30$ years. We then assume that at $t_g = 30$ y, all of the c_o people have survived. They form ($c_o / 2$) couples, and each couple instantaneously produces N children, where N is assumed to be a constant.

Used in this way, N is the average number of children per woman, averaged over all women whether they have children or not. This could be called the "overall fertility." The conventional use of the term "fertility" refers to the average number of children per woman, averaged only over the women who have children. Thus the overall fertility N is slightly smaller than the fertility. For a population in which half of those born are male and half are female, the condition for stability is that the "overall fertility," N, must equal 2.0 , while the "fertility" must equal approximately 2.1 .

There are now a total of $c_1 = (c_o / 2) N$ children in the first cohort. At $t = 60$ y, the c_1 children are all assumed to have survived and are able to reproduce. They form ($c_o / 2$) ($N / 2$) couples and each couple instantaneously produces N children. Thus the total number of children in this second cohort is $c_2 = c_o (N / 2)^2$. The number of children born as the nth cohort is:

$$c_n = c_o (N / 2)^n \tag{14}$$

In this case, the size of successive cohorts is given by this sequence of numbers c_1 , c_2 , c_3 , . . . c_n .

However, the size of the population at any time is the sum of the sizes of the living cohorts. The cs would be the sizes of the population in the case of salmon where the "parents" fertilize the eggs and then die without ever seeing their "children." In the case of humans, several cohorts are living at the same time, so the growth of populations is described by the growth of sums of cs , one for each of the living cohorts. This is illustrated in Table 3 where the sums of cs are tabulated that give the total sizes of populations for the cohorts listed in column 1. In column 2, only one cohort is living at any time and parents never see their children. In column 3, two cohorts are living at any time; parents raise their children, but die when the grandchildren are born. In column 4, three cohorts are living at any time; parents die when their great grandchildren are born.

In order to calculate the FR of the population with the use of Eq. 1, we have to know the ratio of the sizes of the population P for two successive cohorts, one generation time apart. We use Eq. 14 to calculate the values

TABLE 3

The growth of populations when one, two, or three cohorts are living at the same time. The values of the size P_n of the population are tabulated as sums of C's.

Column 1 is the number of the generation

Column 2 is the size of populations in the case where the individuals die when their children are born.

Column 3 is the size of populations in the case where the individuals die when their grandchildren are born.

Column 4 is the size of populations in the case where the individuals die when their great grandchildren are born.

The values of the cs below are, $c_n = c_o(N/2)^n$ where N is the number of children per couple.

1	2	3	4
0	c_o	c_o	c_o
1	c_1	$c_o + c_1$	$c_o + c_1$
2	c_2	$c_1 + c_2$	$c_o + c_1 + c_2$
3	c_3	$c_2 + c_3$	$c_1 + c_2 + c_3$
4	c_4	$c_3 + c_4$	$c_2 + c_3 + c_4$
5	c_5	$c_4 + c_5$	$c_3 + c_4 + c_5$
6	c_6	$c_5 + c_6$	$c_4 + c_5 + c_6$
n	c_n	$c_{(n-1)} + c_n$	$c_{(n-2)} + c_{(n-1)} + c_n$

of each of the cs in Table 3. Notice that, because of startup problems, the equilibrium population pattern is not reached in column 3 until the arrival of the first cohort, and is not reached in column 4 until the arrival of the second cohort.

Let us assume that there are three cohorts living at any time and let us calculate the size of the population after the birth of the 5th cohort, P_5 using column 4 of Table 3:

$$P_5 = c_3 + c_4 + c_5 \tag{15}$$

If we use Eq. 14 for the values of the cs:

$$P_5 = c_o(N/2)^3 + c_o(N/2)^4 + c_o(N/2)^5 \tag{16}$$

We can factor out the common term:

$$P_5 = c_o (N / 2)^3 [1 + (N / 2) + (N / 2)^2] \quad (17)$$

With three living generations, the population P_n after the birth of the nth cohort is:

$$P_n = c_o (N / 2)^{(n-2)} [1 + (N / 2) + (N / 2)^2] \quad (18)$$

Using Eq. 18, we can demonstrate that the ratio of the size of two successive populations is the same as the ratio of the size of the two successive cohorts:

$$P_n / P_{(n-1)} = c_n / c_{(n-1)} = (N / 2) \quad (19)$$

Important Note: Equation 19 was shown to be true for the example of having three cohorts living simultaneously. The analysis of Eqs. 15 through 18 can be repeated for any assumed number of cohorts living at one time. This gives the result that Eq. 19 is true, independent of the assumed number of cohorts that are living simultaneously.

Thus, Eq. 1 shows that the FR of the population will be:

$$k_p = (1 / t_g) \ln (N / 2) \quad (20)$$

Because the logarithm of a quotient is the difference of the logarithms of the numerator and the denominator, Eq. 20 can be written as:

$$k_p = (1 / t_g) \ln (N) - (1 / t_g) \ln (2) \quad (21)$$

Equation 21 gives us the FR of a population in terms of the generation time and the overall fertility N. Curves from Eq. 21 showing the PR of a population as a function of the overall fertility N are plotted in Figs. 1 and 2 for assumed generation times of 20 to 35 years. As expected, all curves show that the PR is zero for N = 2 children per couple.

We note that for comparable cases of the growing number of descendants (Eq. 4) and growing populations (Eq. 21), the overall fertility N of Eq. 4 is replaced by N / 2 in Eq. 21. The FR of Eq. 4 is k_d (for descendants) and the FR of Eq. 21 is k_p (for populations). Using these, Eq. 21 may be written as:

$$k_p = k_d - (1 / t_g) \ln (2) \quad (22)$$

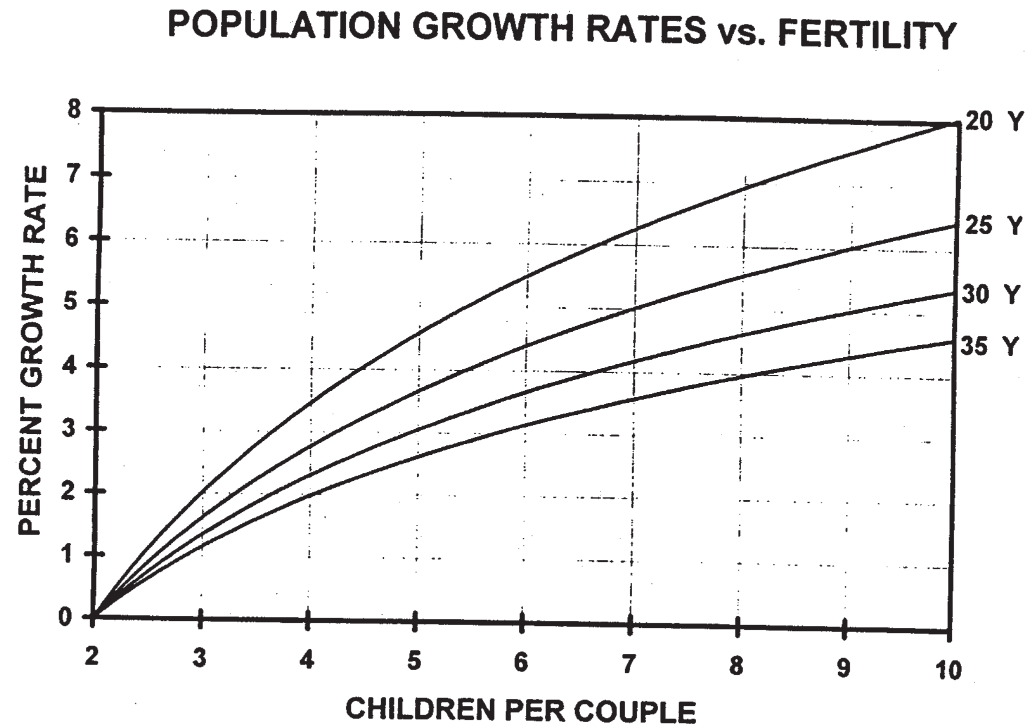

FIGURE 1. The percent growth rate (PR) of a population are plotted as a function of the number of children per couple, for N > 2 and for four different assumed values of the generation time. This plot is prepared using Eq. 20.

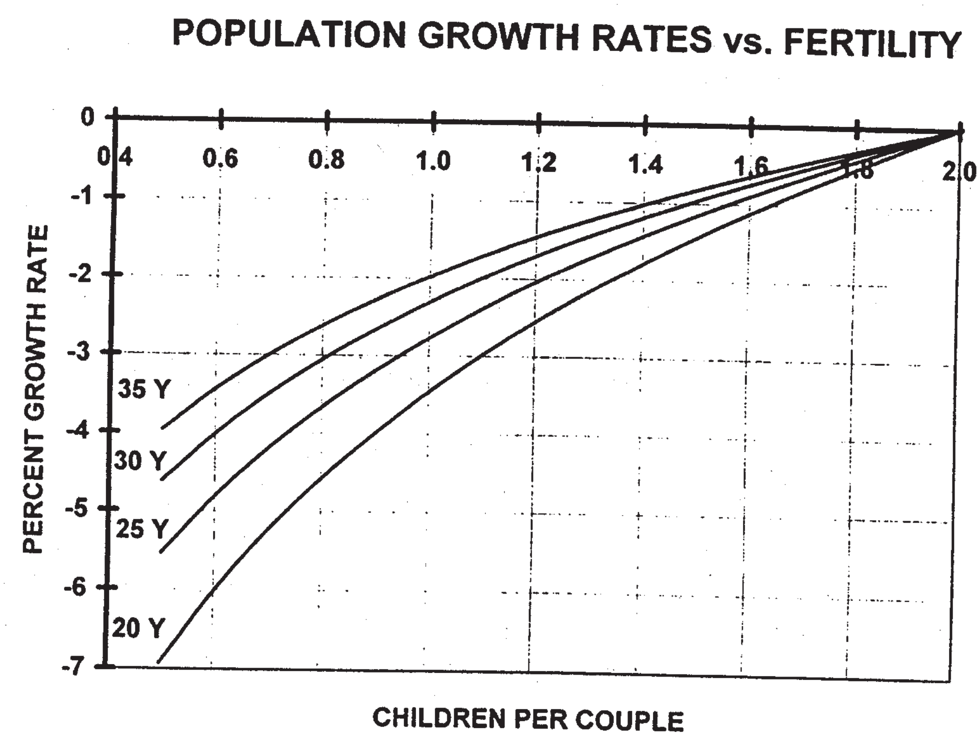

FIGURE 2. This plot shows the same variables as Fig. 1 for the range of N < 2.

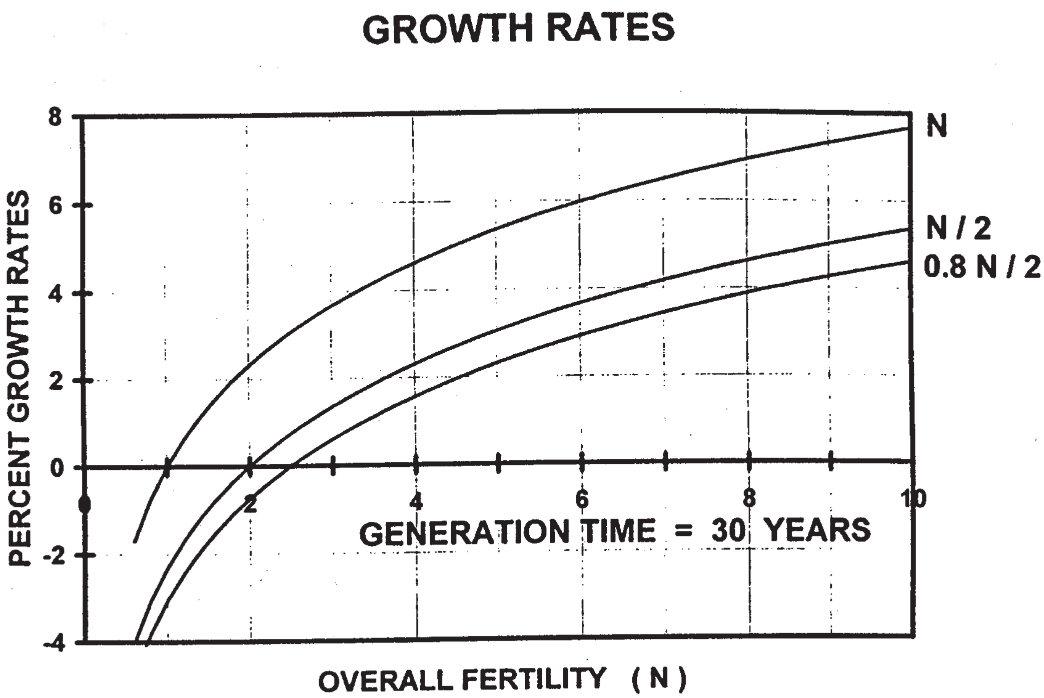

FIGURE 3. For a generation time of 30 years, the upper curve shows, as a function of N, the percent growth rates (PR) of the descendants, all of which are counted whether or not they survive and reproduce. (Eq. 4) The middle curve shows the PR of population where it is assumed that all couples reproduce. (Eq. 11) For this case, the population growth rate is always 2.31% lower than the growth of the rate of descendants. The lower curve shows the PR of population for the case where only 80% of the children in one generation produce members of the following generation: F = 0.8. In this case the population growth rate is always 3.05% lower than the growth rate of descendants.

For a given generation time, the second term is a constant, so that the FR of population growth is a fixed amount lower than the FR of the growth of the number of descendants of a single individual. This relationship is shown in the upper two curves of Fig. 3.

The population model just discussed is based on a constant value of N, just as was our initial model of the descendants of one individual. The results of the more realistic model of descendants resulted only in replacing the constant N of Eq. 4 with the *geometric* mean N_{an} which was defined in Eq. 10 and which replaced the N of Eq. 4 to give Eq. 11. If we are dealing with a population where all the people born in a 30 year generation time are counted as one cohort, we would have to calculate the *arith-*

metic mean number of offspring per *couple* averaged over each cohort one at a time, and then find the *geometric* mean of the numbers determined for each several cohorts. These data are generally available for the small numbers of descendants of one parent, but they are generally not available for larger populations.

It is not clear that we can go farther with this model. The best we can do is to assume the constant overall fertility that leads to Eqs. 20 and 21.

POPULATION GROWTH RATES FROM NUMBERS OF OFFSPRING

We now want to ask: If we know the number of descendants in the nth generation (the More Realistic Model) can we use this datum to calculate k_p , the FR of the population ? (the Population Model)

Equation 21 was Eq. 1 written for two times that are one generation apart, separated by a time t_g . We want to write Eq. 1 for two times that are n generations apart. From Eq. 14:

$$c_n / c_o = (N / 2)^n = N^n / 2^n \qquad (23)$$

The corresponding time interval between c_o and c_n is n t_g . So Eq. 1 becomes:

$$k_p = (1 / n\, t_g) \ln (N^n / 2^n)$$

From Eq. 2, $N^n = C_n$, so we have:

$$k_p = (1 / n\, t_g) \ln (C_n / 2^n) = (1 / T) \ln (C_n / 2^n) \qquad (24)$$

This is the desired relation which allows us to determine the value of k_p if we know the size C_n of the nth cohort of descendants.

Example 5: Data: Find the FR of a population, if all in the population had 67 grandchildren (n = 2).
Calculation 1: We use Eq. 24:

$$k_p = (1 / (2 \times 30)) \ln (67 / 2^2) = 0.047 \text{ or } 4.7\ \% \text{ per year}$$

The reader can consult Figs. 4 and 5 to see the population growth rates that are indicated for various numbers of children, grandchildren and great grandchildren when the generation time is assumed to be 30 years.

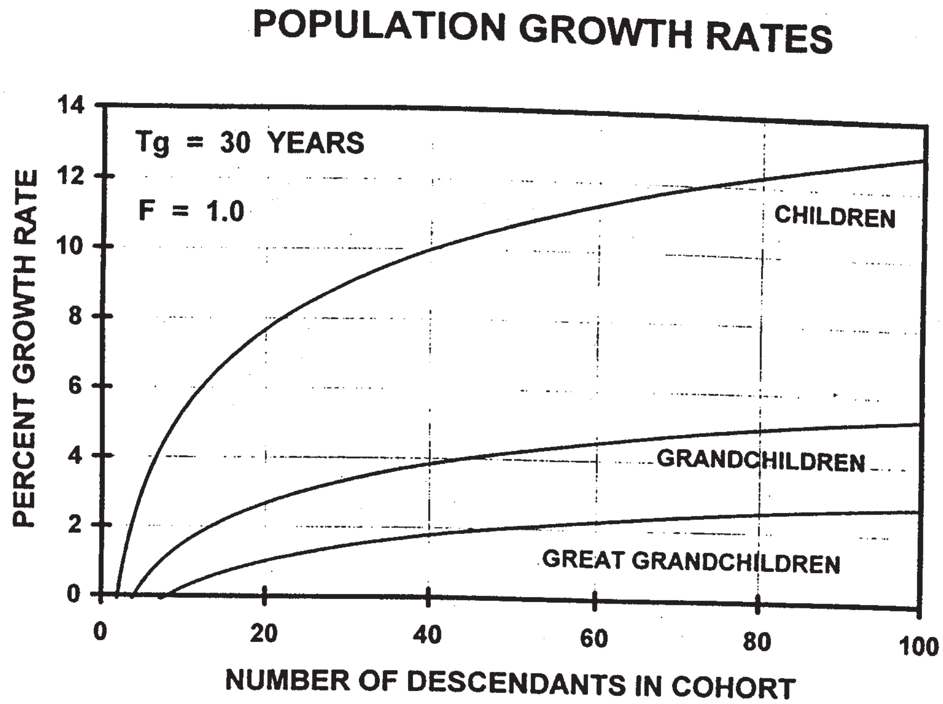

FIGURE 4. This plot shows the PR of a population that is indicated by the numbers of children (top curve), grandchildren (middle curve) and great grandchildren (lower curve). The generation time is assumed to be 30 years. For example, if there were 50 grandchildren, it would indicate a population growth rate of a little over 4% per year. This is a plot of Eq. 24.

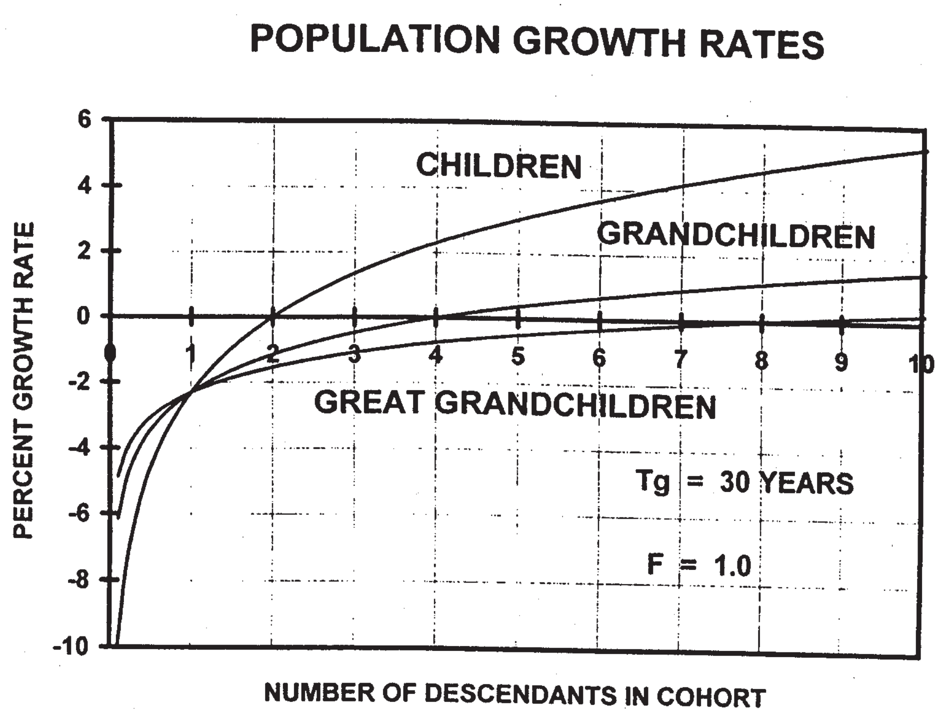

FIGURE 5. This plot shows the same variables as Fig. 4 for the smaller range of values of N. One notes that the population growth rates are zero for the cases of two children, four grandchildren, and eight great grandchildren.

IF OFFSPRING DO NOT ALL REPRODUCE

One can make the population model more realistic if one assumes that, of the children born in a given cohort, only the fraction f_s survive to reproductive age. If c_o children are born at time t = 0, then only f_s c_o survive to reproductive age. At t = 30 years they form f_s c_o / 2 couples. Let us assume that only f_r f_s c_o / 2 of these couples produce offspring. Let us define:

$$F = f_r f_s \tag{25}$$

where F is the fraction of those born in one cohort who survive and who are successful in producing members of the following cohort. At t = 30 y each of the F c_o / 2 couples produces N children. The size c_1 of the first cohort is c_o F N / 2 children. At t = 60 years the second cohort is born, and its size $c_2 = c_o (F N / 2)^2$. In general, at the time (n × 30) years, the size of the newborn nth cohort is:

$$c_n = c_o (F N / 2)^n \tag{26}$$

The ratio of the sizes of successive cohorts or successive populations is, as in Eq. 19:

$$P_n / P_{(n-1)} = c_n / c_{(n-1)} = (F N / 2) \tag{27}$$

As a result, Eq. 1 for k_p gives us an improved expression for the FR :

$$k_p = (1 / t_g) \ln (F N / 2) \tag{28}$$

$$k_p = (1 / t_g) \ln (N) - (1 / t_g) \ln (F / 2) \tag{29}$$

$$k_p = k_d - (1 / t_g) \ln (F / 2) \tag{30}$$

Equation 30 shows that the difference between the FR of population and the FR of descendants increases as F decreases, as is shown in the lower curve of Fig. 3 for which F = 0.8 .

A GENERALIZATION

Equation 11 deals with the growth of the number of descendants of one person; Eq. 20 deals with the growth of the size of a population, and

Eq. 28 deals with the growth of the size of a population when not all persons born in one cohort are able to produce offspring. These three equations are similar. In each case, the argument of the natural logarithm is the ratio of (the number of individuals in the nth cohort who produce offspring), to (the number of similarly productive individuals in the (n − 1) cohort). If we call this ratio the "generational gain," our generalized growth equation is:

$$k = \frac{\ln(\text{generational gain})}{\text{generation time in years}} \tag{31}$$

where the dimensions of k are "fractional gain per year."

It can be seen that these equations are quite the same as Eq. 1. In all of these similar equations we are seeking an "average" FR , where "average" means, "what constant growth rate will result in the observed changes in the time interval during which the changes were observed ?" In the case of Eq. 1, the time interval can have any value. In the subsequent equations in this paper we are dealing with artificial models that produce "staircase phenomena" in which all members of a cohort are born simultaneously and then an assumed fixed time interval t_g elapses before the arrival of the next cohort. This requires that we use times that are integral multiples of the generation time t_g Whether the growing quantity grows in a staircase fashion or in an irregular fashion, the average is the steady growth rate that gives the same increase as the observed increase in the same period of time.

EXAMPLE

Example 6: What growth rates of descendants and of population am I responsible for?

Calculation 1: I have four children, three of whom are married. The married children have given birth to 3, 0, and 2 children respectively, making a total of five grandchildren, the youngest of whom was born in 1988. (For me, F = 1.0: for my children, F = 0.50 .) On the mean birthdate of my children I was 30.30 years old so, for the n = 1 cohort, the generation time t_g can be taken as 30.30 years. On the mean birthdate of my five grandchildren, I was 59.69 years old, so that the generation time t_g for the n = 2 cohort is 29.39 years. The generation time averaged over the two generations is 59.69 / 2 = 29.85 years.

Calculation 2: The n = 1 cohort: The calculation of the FR of my descendants, based on my number of children, is the same whether one uses Eqs. 11, 12, or 13 . From Eq. 12:

$$k_d = (1 / 30.30)(1 / 1)\ln(4) = 0.046 \text{ or } 4.6\ \% \text{ per year}$$

The calculation of the FR of the population growth, based on my number of children is:

$$k_p = (1 / 30.30)(1 / 1)\ln(4 / 2) = 0.023 \text{ or } 2.3\ \% \text{ per year.}$$

Calculation 2: The n = 2 cohort: My four children produced a total of five grandchildren, so the average number of grandchildren per child is (5 / 4). For the n = 2 cohort, the FR of my descendants is:

$$k_d = (1 / 29.39)(1 / 1)\ln(5 / 4) = 0.0076 \text{ or } 0.76\ \% \text{ per year}$$

For the n = 2 cohort, the FR of the population growth is:

$$k_p = (1 / 29.39)(1 / 1)\ln((5 / 4)(1 / 2)) = -0.016 \text{ or negative } 1.6\ \% \text{ per y}$$

Calculation 3: the n = 1 and n = 2 cohorts combined: In order to average over the two generations, it is necessary to find N_1 and N_2 . The number of my children is $N_1 = 4$. The average number of grandchildren per child is $N_2 = C_2 / N_1 = 5 / 4 = 1.25$. For the two cohorts, the *geometric* mean number of offspring per person in the previous cohort is:

$$N_{a2} = (4 \times 1.25)^{1/2} = 2.24$$

So if I had had 2.24 children and each of them had had 2.24 children, there would now be 5 grandchildren. The calculations of k_d (descendants) based on the number of grandchildren, can be done using Eqs. 11, 12, or 13:

(11) $k_d = (1 / 29.85)\ln(2.24) = 0.027$ or 2.7 % per year

(12) $k_d = (1 / 29.85)(1 / 2)\ln(5) = 0.027$ or 2.7 % per year

(13) $k_d = (1 / 59.69)\ln(5) = 0.027$ or 2.7 % per year

As expected, the three calculations give the same result.

The calculation of k_p (population) based on the data for the two generations is, from Eq. 24:

$$k_p = (1 / 59.69)\ln(5 / 2^2) = 0.0037 \text{ or } 0.37\ \% \text{ per year}$$

THE CONDITION FOR STABILITY

The number of descendants will not change from one cohort to the next, only if $k_d = 0$. This is achieved if N of Eq. 4 or C_n of Eqs. 5 or 6 has the value 1.00. This means that in each cohort, the number of descendants of a single parent is unity. Since the offspring were not produced by one parent or ancestor, but by two, one half the descendants in a real family can be attributed to the father and one half to the mother who are the zeroth cohort. Thus, when this father and mother each have one descendant in the nth generation or cohort, the population will be stable.

For the growth of populations, the overall fertility N must have the value 2.0 in Eq. 20 in order to achieve a stable non-growing population.

But not all individuals survive, marry, and have children. For the more realistic situation where there are losses between generations, Eq. 28 shows that the condition for stability of the population is:

$$F\,N / 2 = 1 \qquad (32)$$

It has been observed that the number of children per woman for those women that have children (the fertility) must be approximately 2.1 , in order to have a constant population. This would imply that the product (f_r f_s = F) has the value (2 / 2.1 = 0.95) , which suggests that only about 95 % of those born in a given cohort survive and reproduce.

From this it follows that the condition for a stable population can be represented by the number of descendants in the nth generation for those that reproduce:

$$C_n = (2.1)^n \qquad (33)$$

The sizes of the successive cohorts of descendants of a couple that will maintain a stable population are shown in Table 4.

THE NEWS STORY

The person in the news story is said to have had 67 grandchildren and 201 great grandchildren. Let us assume a generation time, $t_g = 30$ years. We will first do a calculation of the growth of the number of descendants based on the grandchildren ($n = 2$; $C_2 = 67$) . Using Eq. 12:

$$k_d = (1 / 30) (1 / 2) \ln (67) = 0.070 \text{ or } 7.0 \text{ \% per year}$$

TABLE 4

This table gives the average number of descendants in successive generations for a stable population for the case where the fertility rate must be 2.1 children per woman.

Children	$2.1^1 = 2.1$
Grandchildren	$2.1^2 = 4.4$
G Grandchildren	$2.1^3 = 9.3$
GG Grandchildren	$2.1^4 = 19.4$
GGG Grandchildren	$2.1^5 = 41$
Etc.	

The calculation of the population growth rate can be based on the number of grandchildren using Eq. 24:

$$k_p = (1/30)(1/2)\ln(67/2^2) = 0.047 \text{ or } 4.7\ \% \text{ per year}$$

We now do the calculation of the rate of increase of descendants based on the reported number of 201 great grandchildren; ($n = 3$; $C_3 = 201$) :

$$k_d = (1/30)(1/3)\ln(201) = 0.059 \text{ or } 5.9\ \% \text{ per year}$$

Based on the number of great grandchildren, ($n = 3$) the calculation of the FR of the population is:

$$k_p = (1/30)(1/3)\ln(201/2^3) = 0.036 \text{ or } 3.6\ \% \text{ per year}$$

The growth rate of descendants based on the number of great grandchildren is less than the rate based on the number of grandchildren. This suggests that the fertility of the grandchildren has dropped from that of the children, or not all of the great grandchildren had yet arrived.

We can calculate N_a2 , the *geometrical* average number of children per person based on two cohorts and then based on three. Using the outside terms of Eq. 10, we have for these two calculations:

$$N_{a2} = (67)^{(1/2)} = 8.19 \text{ children per person}$$

$$N_{a3} = (201)^{(1/3)} = 5.86 \text{ children per person}$$

If fertilities remain unchanged, the number of children in the nth cohort is found by using the outside terms of Eq. 9 . If the fertility that produced 67 grandchildren remains at 8.19 children per person for following generations, the number of descendants in coming generations will be

$$(n = 3)\ C_3 = (8.19)^3 = 548$$

$$(n = 4)\ C_4 = (8.19)^4 = 4489$$

$$(n = 5)\ C_5 = (8.19)^5 = 36{,}744$$

Similar calculations can be done using N_{a3} .

These results are summarized in Tables 5.

TABLE 5

This table summarizes the number of descendants in each cohort for the person who has 67 grandchildren (column 2) or who has 201 great grandchildren (column 3), assuming t_g = 30 years. The numbers are rounded down to the nearest integer. The numbers in brackets are the data on which the calculations are based.

1	2	3	
Cohort	C_2 = [67]	C_3 = [201]	
Number	N = 8.19	N = 5.86	
Zero	1	1	
First	8	6	
Second	[67]	34	
Third	548	[201]	
Fourth	4489	1177	
Fifth	36,744	6897	
For t_g = 30 y,			
k_d = (per y)	0.0700	0.0589	Descendants
Percent per y	7.0 %	5.9 %	
Doubling time	9.9 y	11.8 y	
k_p = (per y)	0.047	0.036	Population
Percent per y	4.7 %	3.6 %	
Doubling time	15 y	19 y	

SUMS OF PEOPLE

It is possible to find the total number of individuals, parent plus all descendants, through the nth cohort. The sum of a geometric series is:

$$S = (L R - A) / (R - 1) \tag{34}$$

where A is the first term, L is the last term, and R is the common ratio of successive terms. For the case of Example 2 in "Simple Model" we had $A = 1$, $L = C_4 = 81$, and $R = 3$:

$$S = (C_4 N - 1) / (N - 1) = (81 \times 3 - 1) / (3 - 1) = 121$$

Thus the total number of descendants in the chain from the one parent through the fourth cohort of 81 people is 121. Equation 34 can be applied to the other cases discussed here.

SUMMARY OF RESULTS

Figure 3 has three curves, all of which are for a generation time of $t_g = 30$ years. The upper curve shows the PR of the number of descendants as given by Eq. 11. The middle curve shows the PR in the population from Eq. 20, and the lower curve shows the PR in the population when the fraction of a cohort that survive and reproduce is $F = 0.8$ as plotted from Eq. 28.

Figure 6 shows the PR from Eq. 28 for values of N greater than 2.1 for the case where F has the value of 0.95. It shows the PR vs. N for four different values of t_g . Figure 7 is an enlarged view of the fertility range from $N = 2$ to $N = 4$ for the generation times of 25 and 30 years, along with data from the U.S. National Center for Health Statistics. (Wattenberg 1984) The visible agreement between the curves and the data suggest that the generation time in the U.S. in the period from 1930 to 1980 is 27 or 28 years. Figure 8 is a plot for values of N that are less than the replacement level of 2.1 children per woman.

POPULATION MOMENTUM

It should be noted that, because of population momentum, the achievement of a given fertility rate does not guarantee the instantaneous

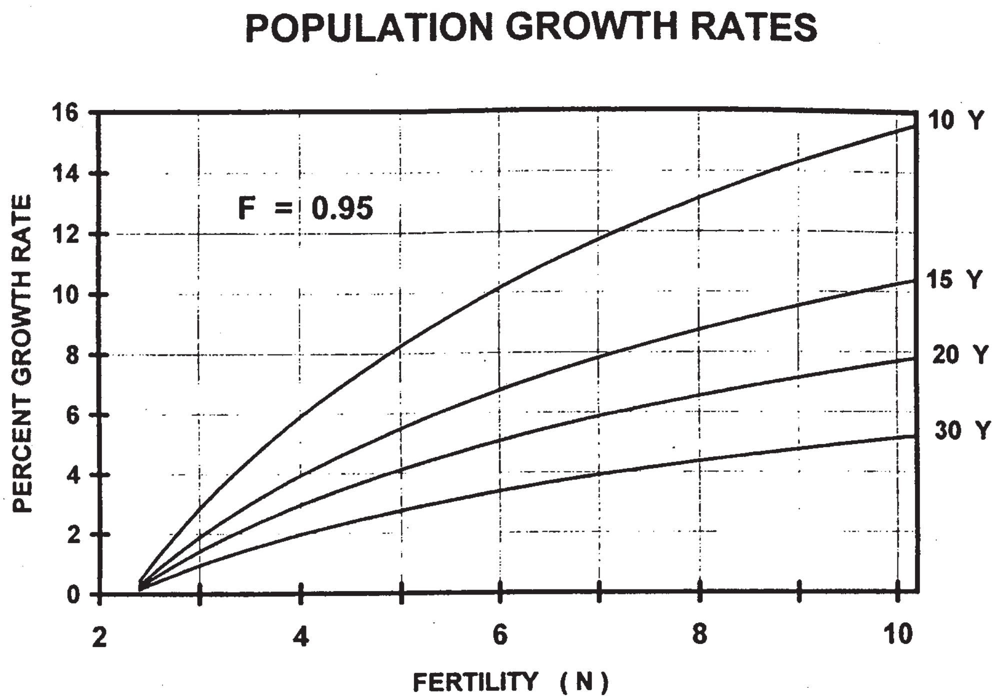

FIGURE 6. Here are plots of the values of the PR of the population (Eq. 28) as a function of N for values of N > 2.1 for four values of t_g (the average generation time). It is assumed that F has the value 0.95.

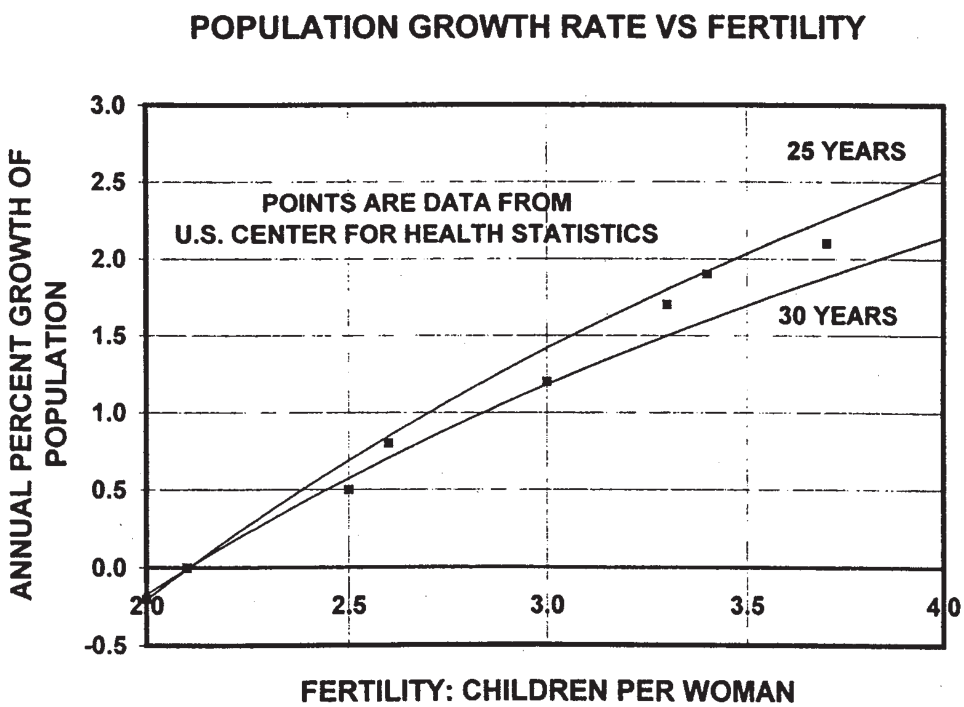

FIGURE 7. This shows an enlarged view of the plot of Fig. 6 in the low fertility range for generation times of 25 years and 30 years. The data points are from the U.S. Center for Health Statistics. (Wattenberg 1984) The agreement between the data point and the curves suggests the generation time in the U.S. in the period 1930–1980 was approximately 27 or 28 years.

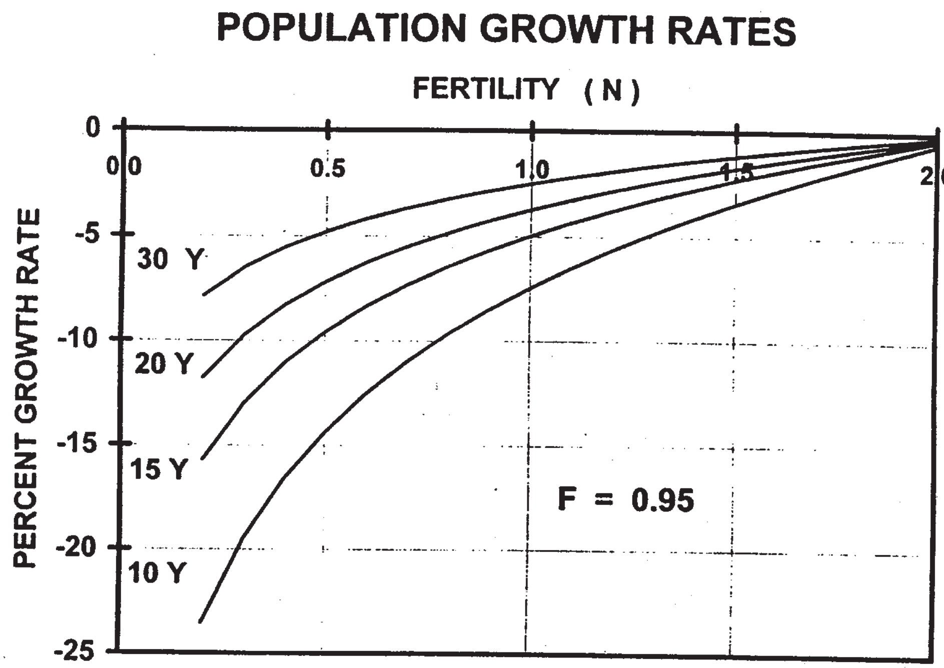

FIGURE 8. This is a plot of the same variables as in Figs. 6 and 7, except that it is for values of N < 2.1. For example, in this figure we see that for t_g = 20 y, N = 0.8, and if 0.95 children per family reproduce successfully, one will have a population whose number decrease approximately 5.0% per year.

achievement of the corresponding population growth rate. The origin of population momentum lies in the fact that in a society with no immigration or emigration, when a fertility rate changes from one value to a new value, *the full effect of the change will not be seen until every individual has died who was living when the change was made,* which is a period of approximately 70 years (Bartlett & Lytwak, 1995). During that approximately 70 year period the growth rate will gradually approach the new rate, reaching the new rate at the end of the 70 years.

The government of the People's Republic of China has set the goal of one child per family, which corresponds roughly to a fertility of one. If F = 0.95, if the generation time is 30 years, and if the fertility is 1.0 child per woman, the expected equilibrium growth rate from Fig. 8 will be approximately a negative 2.5 % per year. This equilibrium growth rate will be reached only after the policy has been in effect for 70 years, providing that throughout this 70 years there would be no net migration. If these conditions are maintained, then throughout the 70 years the population growth rate would gradually make the transition from the condition of growth to its final declining value of minus 2.5 % per year.

ERRORS AND UNCERTAINTIES

In the real world, all members of a cohort are not born simultaneously. People are born every hour of the day and night, and it may not be easy to identify a well-defined generation time, t_g . The models presented here are not very realistic. Significant improvements in the modeling process would probably require a complex computer simulation.

It is not possible to say what the errors are without comparing the results of the model with real-life examples or with the results of a complex computer model. In the two cases from the literature cited below, the results seem reasonable. It is because of these uncertainties that percent growth rates are quoted here only to two significant figures. This implies uncertainties of between 1 % and 10 %. One can guess that the discrepancies between numbers calculated using this model and numbers from the real world would rarely be as small as 1 % or as large as 10 %.

APPLICATION TO TWO CASES

Example 7: Calculate FR for this example of high fertility.
Calculation 1: Hern reports that the Shipibo people of Peru, "have the highest fertility ever recorded for a human group, with a woman having an average of ten births during her reproductive lifetime." (Hern, 1992a) If one uses $t_g = 25$ y as a guess of the age of this mother near the middle of the childbearing years, then Eq. 20 becomes:

$$k_p = (1 / 25) \ln (10 / 2) = 0.064 \text{ per year}$$

This is a growth rate of 6.4 % per year, which is higher than Hern's reported five-year average population growth rate in the Shipibo village of 4.9 % per year. But Eq. 20 assumes that all offspring survive and reproduce, so this difference could be explained if a number of the children die, fail to reproduce, or emigrated away so their offspring are no longer counted in the original population (Hern, 1977).

Calculation 2: We can use Hern's 4.9 % growth rate in Eq. 26 to estimate the fraction $F = f_r f_s$ that fail to survive and to reproduce:

$$0.049 = (1 / 25) \ln (F (10 / 2))$$

Solving for F we have:

$$F = (1/5)\, e^{(0.049 \times 25)} = 0.68$$

Thus Hern's reported population growth rate leads to the estimate that about 68 % of the ten children of the average mother do reach reproductive maturity and reproduce within the observed population. Hern reports that the proportion of living children ranged from 62 % to 70 % among the women in his study (Hern, 1977; 1992b). The predicted 68 % is consistent with the values reported by Hern.

Example 8: Calculate the F for this example.
Calculation 1:

A less extreme example is reported in the Philippines where "the average Philippine woman has 4.1 children during her reproductive lifetime." The population of the Philippines is reported to be growing at an annual rate of 2.5 %. ($k_p = 0.025$) (Nayalkar, 1993) Here we have a case where $N = 4.1$. Let us assume that $t_g = 25$ years. We can use Eq. 26 to estimate the value of F :

$$0.025 = (1 / 25) \ln (F (4.1 / 2))$$

$$F = (2 / 4.1)\, e^{(0.025 \times 25)} = 0.91$$

This suggests that 91 % of those born survive and reproduce.

GROWTH IN THE PRODUCTION OF Ph.D.s

Let us examine growth in another sector of society and address the production of Ph.D.s in higher education. Here we have a kind of parthenogenesis. The individual Ph.D. research professor can produce new Ph.D.s without the need for a "spouse." The thesis professor is the lone "parent" and the students who complete their Ph.D. degrees working with the professor are all "children" of the professor.

In humans, the "generation time" is the time from birth to something like the midpoint of the period of reproductive maturity. The "reproductive life" of a professor in a Ph.D. granting institution is perhaps 30 years. Even though Ph.D. granting professors are reproductively mature "at birth," we will assume that all new Ph.D.s are produced simultaneously at the end of the assumed 30 year productive life. The arithmetic is the same as in the earlier discussion of the growth of the number of descendants of one parent.

Let N be the average number of new Ph.D.s produced by one professor in that professor's "reproductive lifetime." Some of the new Ph.D.s enter academic positions in Ph.D. granting institutions where they can be academically reproductively fertile. Other Ph.D.s elect to be academically celibate and to take positions in undergraduate colleges, government, business, or industry where they do not produce new Ph.Ds. Let f be the fraction of the Ph.D.s who go into graduate-level academic institutions where they are expected to produce more Ph.D.s.

Let us start with one new Ph.D. professor. We will call this person the zeroth cohort. At the end of the professor's career, at $t = 30$ y, the professor instantaneously produces N new Ph.D.s who can be thought of as being "children" of the professor. These children are the first cohort. Of this first cohort, the number $f N$ return to Ph.D. granting institutions where they are fertile, while the remaining $(1 - f) N$ choose academic celibacy. At $t = 60$ y, these $f N$ new professors each instantaneously produce N new Ph.D.s. There are now $f N^2$ new Ph.D.s in the second cohort who are academic "grandchildren" of the original professor. In this second cohort $(f N)^2$ of the Ph.D.s return to academe, and $(1 - f) f N^2$ enter non-reproductive professions. The development of successive cohorts of this population is shown in Table 6.

TABLE 6

Population Growth of Ph.D.s

Gen. # Cohort	Years	# Fac.	#Ph.D.s Produced	# Fertile	# Celibate
	At $t = 0$ years	0			
0	0 to 30	1			
	At $t = 30$	1	N		
1	30 to 60	$f N$		$f N$	$(1 - f) N$
	At $t = 60$	$f N$	$f N^2$		
2	60 to 90	$(f N)^2$		$(f N)^2$	$(1 - f) f N^2$
	At $t = 90$	$(f N)^2$	$f^2 N^3$		
3	90 to 120	$(f N)^3$	$(f N)^3$	$(1 - f) f^2 N^3$	
	At $t = 120$	$(f N)^3$	$f^3 N^4$		
4	120 to 150	$(f N)^4$	$(f N)^4$	$(1 - f) f^3 N^4$	
Etc.					

After the retirement of the first professor, the ratio of the size of the nth cohort to that of the (n − 1)st cohort for the academic population and also for the celibate population, is f N . The FR of each of these populations is then:

$$k = (1 / t_g) \ln (f N) \tag{35}$$

For example, if N = 18 Ph.D. students per professor in the professor's lifetime, and if f = 0.25 (25 % of the Ph.D.s choose reproductive careers):

$$k = (1 / 30) \ln (0.25 \times 18) = 0.050 \text{ per year or } 5.0\ \% \text{ per year}$$

This corresponds to a doubling time T_2 of 14 y. In a professor's productive lifetime of 30 years, the population of Ph.D.s will more than quadruple!

This is a very large growth rate. If it is representative, then what is now referred to as the "overproduction of Ph.D.s" in many academic areas can be seen to be completely and easily predictable.

There have been reports of professors "producing" over 100 Ph.D.s in their academic lifetime. If only one tenth of these enter the academic professions, we will have a growth rate k :

$$k = (1 / 30) \ln (0.1 \times 100) = 0.077 \text{ or } 7.7\ \% \text{ per year}$$

This annual growth rate has a doubling time of about 9 years, and in 30 years would increase the number of Ph.D.s by about a factor of ten.

Figure 9 shows the annual growth rate of Ph.D.s as a function of the number of Ph.D.s produced by one professor for several values of f and for an assumed generation time of 30 years.

If all Ph.D.s returned to positions in Ph.D. granting institutions, then the condition for stability of the population of Ph.D.s would be to have each professor in a Ph.D. granting institution produce only one new Ph.D. in his / her entire career. In the situation where there are two avenues of employment for Ph.D.s, one can indicate a condition for stability in each avenue. The condition for a stable population of Ph.D. granting professors is that the quantity (f N) = 1 . For example, if f = 0.1 and if professors each produce ten Ph.D.s, then one new Ph.D. will replace a professor, and nine will go out into lives of academic celibacy. This will lead to a rapid growth of the celibate Ph.D.s. If one wants the celibate population to be constant, it is necessary that (1 − f) N = 1 . The only way to satisfy the condition of simultaneous stability of the academic and non-academic populations of Ph.D.s is for each professor to produce just two new Ph.D.s

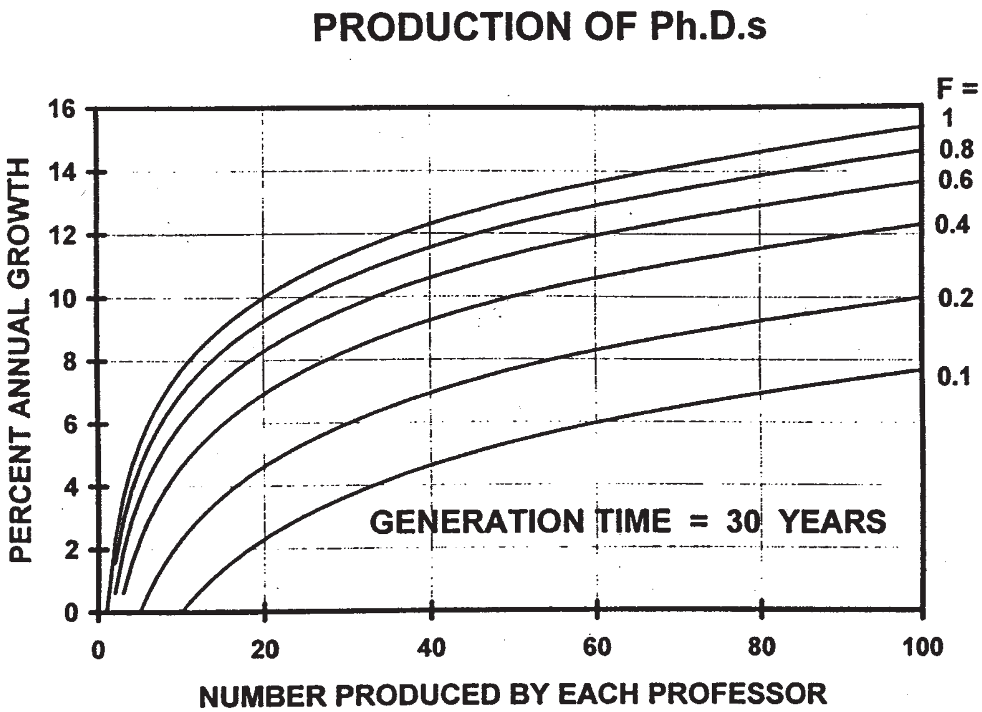

FIGURE 9. This shows the rate of growth of the population of Ph.D.s as a function of the number of Ph.D.s produced by each professor, for several different values of the fraction f of the Ph.D.s that are reproductively fertile; Eq. 34. It is assumed that the generation time is 30 years. For example, if professors produce 50 Ph.D.s and if 40% of them are academically fertile, the population of Ph.D.s will grow 10% per year.

in a lifetime, with one of these replacing the professor in a Ph.D. granting institution, and with the other replacing one in the celibate population. This is satisfied only if $N = 2$ and if $f = 0.5$.

No group of Ph.D.s could have made these Ph.D. growth calculations with greater ease or confidence than the physical scientists. Yet some of physical scientists are now deploring the "shortage of positions" for their newly minted Ph.D.s rather than deploring "longage" of the production of Ph.D.s.

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